THE MULTIPLE HYBRID BOOTSTRAP - RESAMPLING MULTIVARIATE LINEAR PROCESSES

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Abstract. The paper reconsiders the autoregressive aided periodogram bootstrap (AAPB) which has been suggested in Kreiß and Paparoditis (2003). Their idea was to combine a time domain parametric and a frequency domain nonparametric bootstrap to mimic not only a part but as much as possible the complete covariance structure of the underlying time series. We extend the AAPB in two directions. Our procedure explicitly leads to bootstrap observations in the time domain and it is applicable to multivariate linear processes, but agrees exactly with the AAPB in the univariate case, when applied to functionals of the periodogram. The asymptotic theory developed shows validity of the multiple hybrid bootstrap procedure for the sample mean, kernel spectral density estimates and, with less generality, for autocovariances.

1. Introduction

In 1979, Efron's seminal paper on the i.i.d. bootstrap as an extension of the jackknife initiated the fruitful theory of resampling methods in statistics. Since then, a great many of papers concerning resampling techniques for i.i.d. as well as for non i.i.d. data has been proposed, whereas, by now, the i.i.d. case has been understood quite well. However, bootstrap methods have been acknowledged as a powerful tool for approximating certain distributional characteristics of statistics as, for example, variance or covariance, which are sometimes difficult to compute or even not possible to derive analytically. In particular, in time series analysis, due to the potentially complicated dependency structure of the data, often bootstrap methods are required to overcome this barrier, especially, if one wants to avoid the assumption of Gaussianity.

Besides parametric methods that are just applicable to a nonsatisfying narrow class of time series models, several nonparametric approaches for resampling dependent data have been suggested. For instance, Künsch (1989) introduced the so-called block-bootstrap, where blocks of data from a stationary process are resampled to preserve the dependency structure to some extent. See Bühlmann (2002), Lahiri (2003) and Härdle, Horowitz and Kreiß (2003) for an overview of existing methods.

In recent years, bootstrap procedures in the frequency domain have become more and more popular [compare Paparoditis (2002) for a survey]. Most of them are based on resampling the periodogram like in the paper by Franke and Härdle (1992), who proposed a nonparametric residual-based bootstrap that uses an initial (nonparametric) estimate of the spectral density and i.i.d. resampling of (appropriately defined) frequency domain residuals. They proved asymptotic validity for kernel spectral density estimates while Dahlhaus and Janas (1996) extended these validity to ratio statistics and Whittle estimators. Paparoditis and Politis (1999) followed an alternative approach exploiting smoothness properties of the spectral density and resample

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locally from adjacent periodogram ordinates. In an early unpublished manuscript Hurvich and Zeger (1987) use the property that the relation between periodogram and spectral density can be described by means of a multiplicative regression model.

The idea of Kreiß and Paparoditis (2003) was to combine a time domain parametric and a frequency domain nonparametric bootstrap to widen the class of periodogram statistics for which their autoregressive aided periodogram bootstrap (AAPB) remains valid. They use a parametric (autoregressive) fit to catch the essential features of the data and to imitate the weak dependence structure of the periodogram ordinates while a nonparametric correction (in the frequency domain) is applied in order to catch features not represented by the parametric fit.

However, the above mentioned frequency based resampling procedures share one handicap. All of them generate bootstrap periodogram replicates and, for this reason, can be applied to statistics that are functionals of the periodogram, exclusively.

In this paper, we pick up the idea of the AAPB bootstrap introduced by Kreiß and Paparoditis (2003) and enhance their method in two directions. On the one hand, we modify the AAPB in such a manner that our new procedure has the ability to provide explicitly bootstrap replicates in the time domain. Further, we generalize our approach to the multivariate case, on the other hand.

So far, there is only little literature on bootstrap for multivariate time series, especially on nonparametric bootstrap methods. However, one dimension is evidently not enough to study the possibly sophisticated interdependencies between two or more quantities measured over time. Particularly, in econometric work, interest often centers on cross-variable dynamic interactions, which are frequently described with the concept of cointegration. For instance, in the case of a univariate linear time series, the empirical autocovariances concerning different lags obey a CLT with specific handsome covariance matrix in the limit [see Brockwell & Davis (1991), Proposition 7.3.1]. For this reason, using the Δ-method, the limiting covariance matrix of the empirical autocorrelations is not affected by the fourth moments of the white noise process. This fact, in turn, allows the AAPB to be valid for autocorrelations and for ratio statistics in general. If one considers multivariate linear time series this does not remain true any longer. Compare Hannan (1970, chapter IV, section 3 and Theorem 14, p. 228) for the unattractive shape of the entrywise asymptotic covariance structure. Here, bootstrap methods may help approximating the distribution of these statistics.

Paparoditis (1996) considered a parametric bootstrap for vector-valued autoregressive time series of infinte order. The approach of Franke and Härdle (1992) has been extended to the multivariate case by Berkowitz and Diebold (1997) without proving validity. Dai and Guo (2004) proposed to smooth the Cholesky decomposition of a raw estimate of a multivariate spectrum, allowing different degrees of smoothness for different elements, while Guo and Dai (2006) extended their method to multivariate locally stationary processes. Goodness-of-fit tests for VARMA models are investigated by Paparoditis (2005), where the asymptotic distribution of the test statistic is established and therefor a bootstrap method is developed.

In the following we prove validity of our multiple hybrid bootstrap method under some mild general assumptions for the sample mean and for kernel spectral density estimators as well as asymptotic normality for empirical autocovariances, where the here proposed method is shown to work in some important special cases. Moreover, we check the validity for some statistics deduced from the above mentioned as, for example, quadrature and co-spectrum.

The paper is organized as follows. In Section 2, at first, we discuss our idea how to extend the AAPB to get bootstrap observations in the time domain and, thereafter, we generalize this concept to the multivariate case. The technical assumptions needed throughout the paper are summarized in Section 3 while the multiple hybrid bootstrap procedure is described in detail in Section 4. Section 5 deals with applications of the suggested bootstrap in approximating the sampling behavior of sample mean, spectral density estimates and empirical autocovariances as well as from these quantities deduced statistics. Proofs of the main results as well as of some technical lemmas are found in Section 6. Finally, in Section 7 some small simulation studies are presented.

2. Preliminaries

We consider a strictly stationary r-dimensional process $\underline{\mathbf{X}} = (\underline{X}_t : t \in \mathbb{Z})$ and assume that $\underline{X}_t = (X_{t,1}, \ldots, X_{t,r})^T$ has the linear representation

$$\underline{X}_t = \sum_{\nu=-\infty}^{\infty} \mathbf{C}_\nu \underline{\epsilon}_{t-\nu}, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $\mathbf{C}_\nu = (C_{\nu,ij})_{i,j=1,\ldots,r}$, $\nu \in \mathbb{Z}$ are real $(r \times r)$ matrices, $\mathbf{C}_0 = \mathbf{I}_r$ is the $(r \times r)$ unit matrix and the sequence $(\mathbf{C}_\nu : \nu \in \mathbb{Z})$ is entrywise absolutely summable. Further, the error process $(\underline{\epsilon}_t : t \in \mathbb{Z})$ is assumed to consist of r-dimensional independent and identically distributed random variables $\underline{\epsilon}_t = (\epsilon_{t,1}, \ldots, \epsilon_{t,r})^T$ with $E[\underline{\epsilon}_t] = \underline{0}$ and $E[\underline{\epsilon}_t \underline{\epsilon}_t^T] = \mathbf{\Sigma}$, where the $(r \times r)$ covariance matrix $\mathbf{\Sigma}$ is supposed to be positive definite. Under these assumptions, $\underline{\mathbf{X}}$ exhibits the spectral density

$$\mathbf{f}(\omega) = \frac{1}{2\pi} \left(\sum_{\nu=-\infty}^{\infty} \mathbf{C}_\nu e^{-i\nu\omega} \right) \mathbf{\Sigma} \overline{\left(\sum_{\nu=-\infty}^{\infty} \mathbf{C}_\nu e^{-i\nu\omega} \right)}^T. \tag{2.2}$$

Here and in the following, we underline vector-valued quantities and write matrix-valued ones as bold letters. $\overline{\mathbf{Z}}$ denotes the (entrywise) complex conjugate of a matrix $\mathbf{Z}$ and X^T indicates the transpose of a vector or matrix X.

Since our first main intention is to pick up the concept of the AAPB bootstrap proposed by Kreiß and Paparoditis (2003) and modify it to obtain a procedure that is explicitly able to generate bootstrap replicates in the time domain, initially, we consider the univariate case $r = 1$ to simplify matters and sketch the steps of their method before demonstrating which step is the sticking point.

The univariate AAPB approach can be summarized as follows. With real-valued observations $X_1, \ldots, X_n$ at hand, Kreiß and Paparoditis apply an usual residual-based autoregressive bootstrap of fixed order $p \in \mathbb{N}$ to obtain bootstrap replicates $X_1^+, \ldots, X_n^+$ and compute the periodogram $I_n^+(\omega) = \frac{1}{2\pi n} |\sum_{t=1}^n X_t^+ e^{-it\omega}|^2$ of these quantities to switch over to the frequency domain. So far, this is just a parametric bootstrap that, of course, is not valid asymptotically if the underlying data does not stem from an autoregressive model of order less or equal to p. Therefore, they *correct* the periodogram $I_n^+(\omega)$ by multiplication with a nonparametric (pre-whitening) correction function $\widehat{q}(\omega)$, defined as

$$\widehat{q}(\omega) = \frac{1}{n} \sum_{j=-N}^{N} K_h(\omega - \omega_j) \frac{I_n(\omega_j)}{\widehat{f}_{AR}(\omega_j)}, \tag{2.3}$$

where $\omega_j = 2\pi\frac{j}{n}$, $N = [\frac{n}{2}]$, h is the bandwidth, K is a kernel function, $K_h(\cdot) = \frac{1}{h}K(\frac{\cdot}{h})$, $I_n(\omega)$ is the periodogram based on $X_1, \ldots, X_n$ and $\widehat{f}_{AR}$ is the spectral density obtained from the autoregressive fit. Their proceeding is motivated by the following facts. Recall that we want to bootstrap the periodogram $I_n(\omega)$ and under some assumptions on the coefficients of the linear representation of X_t in (2.1), it holds

$$E[I_n(\omega)] = f(\omega) + o(1), \tag{2.4}$$

but using the simple residual AR-bootstrap, however, yields

$$E^+[I_n^+(\omega)] = f_{AR}(\omega) + o_P(1), \tag{2.5}$$

where f is the true spectral density of the process $\mathbf{X}$ and f_{AR} is the spectral density of the theoretical autoregressive model of order p that is obtained as n tends to infinity. Note, that $f \neq f_{AR}$ in general. Here, as usual, E^+ denotes the conditional expectation given $X_1, \ldots, X_n$.

Since the estimate $\widehat{q}(\omega)$ in (2.3) converges to $\frac{f(\omega)}{f_{AR}(\omega)}$ in probability under some reasonable assumptions, their self-evident attempt to solve the problem argued in (2.4) and (2.5) is to design *corrected* bootstrap periodogram replicates $I_n^*(\omega)$ according to

$$I_n^*(\omega) = \widehat{q}(\omega)I_n^+(\omega),$$

obtaining

$$E^+[I_n^*(\omega)] = \widehat{q}(\omega)E^+[I_n^+(\omega)] = f(\omega) + o_P(1), \tag{2.6}$$

which, by now, agrees with the expectation in (2.4). Thus, the last equation emphasizes that, in a certain sense, the AAPB does the right correction in the frequency domain. For this reason, one would expect this method to work for all statistics whose asymptotic distributional characteristics can be written in terms of the spectral density. But there are statistics with this property that cannot be written itself by means of the periodogram as, for instance, the sample mean. Recall that under some standard assumptions the following CLT holds true:

$$\mathcal{L}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^n X_t\right) \Rightarrow \mathcal{N}(0, 2\pi f(0)). \tag{2.7}$$

However, using just the simple AR-bootstrap, under suitable assumptions, it holds

$$\mathcal{L}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^n X_t^+|X_1, \ldots, X_n\right) \Rightarrow \mathcal{N}(0, 2\pi f_{AR}(0)) \tag{2.8}$$

in probability. Considering solely (2.7) and (2.8), a naive idea to construct a bootstrap that works for the sample mean is to generate $X_1^+, \ldots, X_n^+$ and multiply the whole data set with $\sqrt{\widehat{q}(0)}$. Doing so, with Slutsky, we get

$$\begin{aligned}\mathcal{L}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^n \sqrt{\widehat{q}(0)}X_t^+|X_1, \ldots, X_n\right) \quad &\Rightarrow \quad \mathcal{N}(0, 2\pi f_{AR}(0)q(0)) \\ &= \quad \mathcal{N}(0, 2\pi f(0))\end{aligned}$$

in probability, but this approach is just taylor-made for the sample mean and does not remain valid in other cases as spectral density estimation or for ratio statistics. Therefore, a different modification of the AAPB has to be developed to solve this problem, but we will come back to this issue later.

Now, to answer the question why the AAPB is not capable to deliver bootstrap replicates in the time domain, observe that $I_n^+(\omega)$, $\omega \in [-\pi, \pi]$ does not contain all the information that is contained in the data set $X_1^+, \ldots, X_n^+$. This means, on the one hand, computing the periodogram causes an irretrievable loss of information, but switching to the frequency domain is necessary to apply the nonparametric correction, on the other hand. To get rid of this inconvenience, note that for the periodogram at the fourier frequencies $\omega_j = 2\pi \frac{j}{n}$, $j = 1, \ldots, n$, it holds

$$I_n^+(\omega_j) = |J_n^+(\omega_j)|^2 = J_n^+(\omega_j)\overline{J_n^+(\omega_j)},$$

where $J_n^+(\omega_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n X_t^+ e^{-it\omega_j}$ is the discrete fourier transform (DFT) and there is an one-to-one correspondence between $X_1^+, \ldots, X_n^+$ and $J_n^+(\omega_1), \ldots, J_n^+(\omega_n)$.

These circumstances result in the idea to compute the DFT $J_n^+(\omega_1), \ldots, J_n^+(\omega_n)$ instead of the periodogram, multiply them with appropriate correction terms $\widetilde{q}(\omega_j)$ and use the ono-to-one correspondence to get back to the time domain. The canonical choice of the correction term is $\widetilde{q}(\omega) = \sqrt{\widehat{q}(\omega)}$ and to set

$$J_n^*(\omega_j) = \widetilde{q}(\omega_j) J_n^+(\omega_j), \quad j = 1, \ldots, n,$$

because with this definition, it holds

$$J_n^*(\omega_j)\overline{J_n^*(\omega_j)} = \widetilde{q}(\omega_j) J_n^+(\omega_j)\overline{\widetilde{q}(\omega_j) J_n^+(\omega_j)} = \widehat{q}(\omega_j) I_n^+(\omega_j) = I_n^*(\omega_j), \tag{2.9}$$

which is exactly the correction done in the AAPB method. Finally, we exploit the one-to-one correspondence of the DFT, to define bootstrap observations $X_1^*, \ldots, X_n^*$ via inverse DFT, that is,

$$X_t^* = \sqrt{\frac{2\pi}{n}} \sum_{j=1}^n J_n^*(\omega_j) e^{it\omega_j}, \quad t = 1, \ldots, n. \tag{2.10}$$

Now, that we have developed a bootstrap method that directly leads to bootstrap observations in the time domain and, moreover, contains the AAPB as a special case, let us consider the sample mean discussed in (2.7) and (2.8) again. Using the replicates defined (2.10), we get

$$\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{t=1}^n X_t^* &= \frac{1}{\sqrt{n}} \sqrt{\frac{2\pi}{n}} \sum_{j=1}^n J_n^*(\omega_j) \sum_{t=1}^n e^{it\omega_j} \\
&= \sqrt{2\pi} J_n^*(0) \\
&= \widetilde{q}(0) \frac{1}{\sqrt{n}} \sum_{t=1}^n X_t^+,
\end{aligned}$$

which is exactly the naive correction suggested earlier to construct a bootstrap that works for the sample mean, but contrary to the previous situation this new approach remains valid in all situations where the AAPB is already shown to work thanks to relation (2.9).

Taking everything into account, the above derived bootstrap constitutes a reasonable modification of the AAPB that is able to produce bootstrap replicates in the time domain and, for this reason, is applicable to a wider class of statistics. We call this proposal the (univariate) *hybrid bootstrap*.

Next, we generalize the hybrid bootstrap to the multivariate case. From now on, the data of interest is supposed to have some arbitrary dimension $r \geq 1$, but to appreciate the main difficulties adapting the univariate proposal derived above, consider the vector-valued case $r \geq 2$, only.

The first step of the hybrid bootstrap generalizes to a usual residual-based vector-autoregressive scheme to obtain $\underline{X}_1^+, \ldots, \underline{X}_n^+$. Further, the periodogram

$$\mathbf{I}_n^+(\omega_j) = \underline{J}_n^+(\omega_j)\overline{\underline{J}_n^+(\omega_j)}^T, \quad j = 1, \ldots, n$$

becomes a hermitian $(r \times r)$-matrix and the (multivariate) discrete fourier transform (mDFT) $\underline{J}_n^+(\omega_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^n \underline{X}_t^+ e^{-it\omega_j}$ is now a r-dimensional column vector. Reconsidering (2.4) and (2.5) in the vector-valued case, it still holds

$$E[\mathbf{I}_n(\omega)] = \mathbf{f}(\omega) + o(1) \tag{2.11}$$

as well as

$$E^+[\mathbf{I}_n^+(\omega)] = \mathbf{f}_{AR}(\omega) + o_P(1), \tag{2.12}$$

with $\mathbf{I}_n(\omega)$, $\mathbf{f}(\omega)$ and $\mathbf{f}_{AR}(\omega)$ according to the univariate case.

Maintaining the property to produce bootstrap replicates in the time domain, consequently, we have to correct the mDFT. Now, this has to be done by multiplication with a suitable $(r \times r)$-matrix $\widetilde{\mathbf{Q}}(\omega_j)$, defining

$$\underline{J}_n^*(\omega_j) = \widetilde{\mathbf{Q}}(\omega_j)\underline{J}_n^+(\omega_j), \quad j = 1, \ldots, n.$$

Similar to the univariate equation (2.9), now, we get

$$\underline{J}_n^*(\omega_j)\overline{\underline{J}_n^*(\omega_j)}^T = \widetilde{\mathbf{Q}}(\omega_j)\underline{J}_n^+(\omega_j)\overline{\widetilde{\mathbf{Q}}(\omega_j)\underline{J}_n^+(\omega_j)}^T = \widetilde{\mathbf{Q}}(\omega_j)\mathbf{I}_n^+(\omega_j)\overline{\widetilde{\mathbf{Q}}(\omega_j)}^T. \tag{2.13}$$

Concerning (2.12), the last relation (2.13) asks for the correction term $\widetilde{\mathbf{Q}}(\omega)$ to converge in probability to its limit $\mathbf{Q}(\omega)$ [Observe the notation differing to the univariate case! For $r = 1$, it holds $\mathbf{Q}(\omega) = \sqrt{q(\omega)}$ instead of $\mathbf{Q}(\omega) = q(\omega)$.], which has to satisfy the equality

$$\mathbf{Q}(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T = \mathbf{f}(\omega) \tag{2.14}$$

to get the analogue result to equation (2.6) obtained in the univariate case, that is,

$$E^+[\mathbf{I}_n^*(\omega)] = \widetilde{\mathbf{Q}}(\omega)E^+[\mathbf{I}_n^+(\omega)]\overline{\widetilde{\mathbf{Q}}(\omega)}^T = \mathbf{f}(\omega) + o_P(1).$$

Now, to answer the question how $\widetilde{\mathbf{Q}}(\omega)$ has to be defined to achieve this property, suppose we knew that $\mathbf{f}(\omega)$ and $\mathbf{f}_{AR}(\omega)$ have some representations

$$\mathbf{f}(\omega) = \mathbf{G}(\omega)\overline{\mathbf{G}(\omega)}^T \quad \text{and} \quad \mathbf{f}_{AR}(\omega) = \mathbf{B}(\omega)\overline{\mathbf{B}(\omega)}^T. \tag{2.15}$$

Then, if the inverse of $\mathbf{B}(\omega)$ exists, it seems self-evident to set $\mathbf{Q}(\omega) = \mathbf{G}(\omega)\mathbf{B}^{-1}(\omega)$, obtaining

$$\mathbf{Q}(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T = \mathbf{G}(\omega)\mathbf{B}^{-1}(\omega)\mathbf{B}(\omega)\overline{\mathbf{B}(\omega)}^T\overline{\mathbf{B}^{-1}(\omega)}^T\overline{\mathbf{G}(\omega)}^T = \mathbf{f}(\omega),$$

and accordingly to construct a nonparametric estimator $\widetilde{\mathbf{Q}}(\omega)$ for this quantity $\mathbf{G}(\omega)\mathbf{B}^{-1}(\omega)$.

If $\mathbf{f}(\omega)$ and $\mathbf{f}_{AR}(\omega)$ are positive definite, their uniquely determined Cholesky decompositions as in (2.15) exist, where $\mathbf{G}(\omega)$ and $\mathbf{B}(\omega)$ have full rank. Thus, we can state $\widetilde{\mathbf{Q}}(\omega)$ in terms of estimates for $\mathbf{f}(\omega)$ and $\mathbf{f}_{AR}(\omega)$.

As in the univariate case, $\mathbf{f}(\omega)$ can be estimated nonparametrically by $\widehat{\mathbf{f}}(\omega)$ via smoothing the periodogram matrix and $\mathbf{f}_{AR}(\omega)$ is estimated by $\widehat{\mathbf{f}}_{AR}(\omega)$, which is obtained from the residual vector AR-bootstrap. Assuming $\mathbf{f}(\omega)$ to be positive definite, then, for sufficiently large sample size n in relation to r, the estimates $\widehat{\mathbf{f}}(\omega)$ and $\widehat{\mathbf{f}}_{AR}(\omega)$ are positive definite in probability. Hence, we can define

$$\widetilde{\mathbf{Q}}(\omega) = \widehat{\mathbf{G}}(\omega)\widehat{\mathbf{B}}^{-1}(\omega),$$

where $\widehat{\mathbf{f}}(\omega) = \widehat{\mathbf{G}}(\omega)\overline{\widehat{\mathbf{G}}(\omega)}^T$ and $\widehat{\mathbf{f}}_{AR}(\omega) = \widehat{\mathbf{B}}(\omega)\overline{\widehat{\mathbf{B}}(\omega)}^T$. Observe also the detailed illustration of this *multiple hybrid bootstrap* proposal in Section 4 and, in particularly, Remark 4.1 on the choice of $\widetilde{\mathbf{Q}}(\omega)$.

3. Assumptions

3.1. The data generation process.

We assume the underlying process $\underline{\mathbf{X}}$ to satisfy the following assumptions:

(A1) $(\underline{X}_t : t \in \mathbb{Z})$ is a $\mathbb{R}^r$-valued linear strictly stationary process

$$\underline{X}_t = \sum_{\nu=-\infty}^{\infty} \mathbf{C}_\nu \underline{\epsilon}_{t-\nu}, \quad t \in \mathbb{Z},$$

where $\mathbf{C}_\nu$, $\nu \in \mathbb{Z}$ are $(r \times r)$ coefficient matrices, $\mathbf{C}_0 = \mathbf{I}_r$ is the $(r \times r)$ unit matrix and for all $j, k = 1, \ldots, r$ the summability condition

$$\sum_{\nu=-\infty}^{\infty} |\nu||C_\nu(j,k)| < \infty$$

holds true. Further, $\sum_{\nu=-\infty}^{\infty} \mathbf{C}_\nu z^\nu$ is supposed to be nonsingular on the unit circle, that is

$$det\left(\sum_{\nu=-\infty}^{\infty} \mathbf{C}_\nu z^\nu\right) \neq 0 \quad \forall z \in \mathbb{C} : |z| = 1.$$

(A2) The error process is assumed to be a *standard white noise* [compare Lütkepohl (2005), p.73], that means $(\underline{\epsilon}_t : t \in \mathbb{Z})$ constitutes a sequence of independent and identically distributed $\mathbb{R}^r$-valued random variables with $E[\underline{\epsilon}_t] = \underline{0}$ and $E[\underline{\epsilon}_t \underline{\epsilon}_t^T] = \mathbf{\Sigma}$, where the covariance matrix $\mathbf{\Sigma}$ is supposed to be positive definite. Further, for $i, j, k, l = 1, \ldots, r$ the expectation $E[\epsilon_{t,i}\epsilon_{t,j}\epsilon_{t,k}\epsilon_{t,l}] < \infty$ exists and $\kappa_4(i,j,k,l)$ denotes the fourth-order cumulant between $\epsilon_{t,i}, \epsilon_{t,j}, \epsilon_{t,k}$ and $\epsilon_{t,l}$.

(A3) The spectral density $\mathbf{f}$ in (2.2) of $\underline{\mathbf{X}}$ is (entrywise) three times continuously differentiable on $[-\pi, \pi]$ and accordingly on the real line, when understood as continuously extended.

3.2. The kernel function.

(K1) K denotes a nonnegative kernel function with compact support $[-\pi, \pi]$. The fourier transform k of K, that is,

$$k(u) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} K(x) e^{-ixu} dx,$$

is assumed to be a symmetric, continuous and bounded function satisfying $k(0) = 2\pi$. Hence, the kernel has the representation

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k(u) e^{-iux} du.$$

Note that $k(0) = 2\pi$ implies that $\frac{1}{2\pi}\int_{-\infty}^{\infty} K(u)du = 1$, while the symmetry of k implies the same property for K.

(K2) The fourier transform k of K satisfies $\int_{-\infty}^{\infty} k^2(u)du < \infty$.

(K3) K is three times continuously differentiable on $[-\pi, \pi]$ and its derivatives fulfill the smoothness condition $K^{(d)}(-\pi) = K^{(d)}(\pi) = 0$ for all $d = 0, 1, 2$.

3.3. The bandwidth.

(B1) $h = h(n) \to 0$ as $n \to \infty$ such that $nh \to \infty$.
(B2) $h = h(n) \to 0$ as $n \to \infty$ such that $(nh^4)^{-1} = O(1)$.
(B3) $h = h(n) \to 0$ as $n \to \infty$ such that $(nh^6)^{-1} = O(1)$.

4. The hybrid bootstrap procedure

In this section, first of all, we describe the multiple hybrid bootstrap motivated in Section 2 in detail and, afterwards, we give a couple of comments on the choice of the correction function $\widetilde{\mathbf{Q}}(\omega)$ and thereby arising difficulties. Moreover, we discuss the special case where no autoregressive model is fitted at all.

Step 1. Given the observations $\underline{X}_1, \ldots, \underline{X}_n$, we fit a vector-autoregressive process of order $p\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ ($VAR(p)$-model). This leads to estimated coefficient matrices $\widehat{\mathbf{A}}_1(p), \ldots, \widehat{\mathbf{A}}_p(p)$ and covariance matrix $\widehat{\mathbf{\Sigma}}(p)$, which are obtained from the multivariate Yule-Walker equations. Consider the estimated residuals

$$\underline{\widehat{\epsilon}}_t = \underline{X}_t - \sum_{\nu=1}^{p} \widehat{\mathbf{A}}_\nu(p)\underline{X}_{t-\nu}, \quad t = p+1, \ldots, n$$

and denote $\widehat{F}_n^c$ the empirical distribution function of the standardized quantities

$$\underline{\widetilde{\epsilon}}_t = \widehat{\mathbf{L}}(p)^{-1}\left(\underline{\widehat{\epsilon}}_t - \frac{1}{n-p}\sum_{s=p+1}^{n} \underline{\widehat{\epsilon}}_s\right), \quad t = p+1, \ldots, n,$$

where

$$\widehat{\mathbf{L}}(p)\widehat{\mathbf{L}}(p)^T = \frac{1}{n-p}\sum_{t=p+1}^{n}\left(\underline{\widehat{\epsilon}}_t - \frac{1}{n-p}\sum_{s=p+1}^{n}\underline{\widehat{\epsilon}}_s\right)\left(\underline{\widehat{\epsilon}}_t - \frac{1}{n-p}\sum_{r=p+1}^{n}\underline{\widehat{\epsilon}}_r\right)^T$$

is the Cholesky decomposition of the covariance matrix of the centered residuals. That is, $\widehat{F}_n^c$ has mean $\underline{0}$ and the unit matrix $\mathbf{I}_r$ as covariance matrix.

Step 2. Generate bootstrap observations $\underline{X}_1^+, \ldots, \underline{X}_n^+$ according to the following vector autoregressive model of order p:

$$\underline{X}_t^+ = \sum_{\nu=1}^{p} \widehat{\mathbf{A}}_\nu(p)\underline{X}_{t-\nu}^+ + \widehat{\mathbf{L}}(p)\underline{\epsilon}_t^+,$$

where $(\underline{\epsilon}_t^+)$ constitutes a sequence of i.i.d. random variables with cumulative distribution function $\widehat{F}_n^c$ (conditionally on the given observations $\underline{X}_1, \ldots, \underline{X}_n$). Now, the time series $(\underline{X}_t^+ : t \in \mathbb{Z})$ has the spectral density

$$\widehat{\mathbf{f}}_{AR}(\omega) = \frac{1}{2\pi}\left(\mathbf{I}_r - \sum_{k=1}^{p}\widehat{\mathbf{A}}_k(p)e^{-ik\omega}\right)^{-1}\widehat{\mathbf{\Sigma}}(p)\left(\left(\overline{\mathbf{I}_r - \sum_{k=1}^{p}\widehat{\mathbf{A}}_k(p)e^{-ik\omega}}\right)^{-1}\right)^T.$$

Thereby, the used multivariate Yule-Walker estimates ensure that $\widehat{\mathbf{f}}_{AR}(\omega)$ is always well defined [cf. Whittle (1963)], that is

$$det\left(\mathbf{I}_r - \sum_{\nu=1}^{p}\widehat{\mathbf{A}}_\nu(p)z^\nu\right) \neq 0 \quad \forall z \in \mathbb{C} : |z| \leq 1.$$

Step 3. Compute the (multivariate) discrete fourier transform (mDFT) of the bootstrap observations $\underline{X}_1^+, \ldots, \underline{X}_n^+$, that is

$$\underline{J}_n^+(\omega_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \underline{X}_t^+ e^{-it\omega_j}, \quad j = 1, \ldots, n$$

at the fourier frequencies $\omega_j = 2\pi \frac{j}{n}$, $j = 1, \ldots, n$. Notice, there is an one-to-one correspondence

$$\underline{X}_1^+, \ldots, \underline{X}_n^+ \quad \leftrightarrow \quad \underline{J}_n^+(\omega_1), \ldots, \underline{J}_n^+(\omega_n).$$

Step 4. Define the nonparametric correction function $\widetilde{\mathbf{Q}}(\omega) = \widehat{\mathbf{G}}(\omega)\widehat{\mathbf{B}}(\omega)^{-1}$, where $\widehat{\mathbf{G}}(\omega)$ and $\widehat{\mathbf{B}}(\omega)$ are obtained via the following Cholesky decompositions (in lower triangular matrix times its transposed complex conjugate):

$$\widehat{\mathbf{B}}(\omega)\overline{\widehat{\mathbf{B}}(\omega)}^T = \widehat{\mathbf{f}}_{AR}(\omega), \tag{4.1}$$

$$\widehat{\mathbf{G}}(\omega)\overline{\widehat{\mathbf{G}}(\omega)}^T = \widehat{\mathbf{B}}(\omega)\left(\frac{1}{n}\sum_{k=-N}^{N} K_h(\omega-\omega_k)\widehat{\mathbf{B}}(\omega_k)^{-1}\mathbf{I}_n(\omega_k)\overline{\widehat{\mathbf{B}}(\omega_k)^{-1}}^T\right)\overline{\widehat{\mathbf{B}}(\omega)}^T, \tag{4.2}$$

whereas $N = [\frac{n}{2}]$, K is a kernel function, $K_h(\cdot) = \frac{1}{h}K(\frac{\cdot}{h})$ and h is the bandwidth. Furthermore, $\mathbf{I}_n(\omega)$ denotes the periodogram of the given observations

$$\mathbf{I}_n(\omega) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} \underline{X}_t e^{-i\omega t}\right|^2.$$

Now, compute the nonparametric estimator $\widetilde{\mathbf{Q}}$ at the fourier frequencies $\omega_j = 2\pi\frac{j}{n}$, $j = 1, \ldots, n$. In doing so, all involved quantities are understood as continuously extended to the real line.

Step 5. At first, compute the mDFT $\underline{J}_n^+(\omega_j)$, $j = 1, \ldots, n$ of the parametrically via residual bootstrap generated observations $\underline{X}_1^+, \ldots, \underline{X}_n^+$ and afterwards apply the nonparametric correction function $\widetilde{\mathbf{Q}}$ to get the *corrected* version of the mDFT, that is

$$\underline{J}_n^*(\omega_j) = \widetilde{\mathbf{Q}}(\omega_j)\underline{J}_n^+(\omega_j), \quad j = 1, \ldots, n.$$

Step 6. According to the inverse mDFT, the bootstrap observations $\underline{X}_1^*, \ldots, \underline{X}_n^*$ are defined as follows:

$$\underline{X}_t^* = \sqrt{\frac{2\pi}{n}} \sum_{j=1}^{n} \underline{J}_n^*(\omega_j) e^{it\omega_j}, \quad t = 1, \ldots, n.$$

Remark 4.1 (On the choice of $\widetilde{Q}(\omega)$)**.**

(i) *As illustrated in* (2.15)*, basically, it is possible to use alternative decompositions e.g. square-root or Cholesky-decomposition in upper times lower triangular matrix. Although Cholesky needs positive definiteness, we choose this decomposition (in lower triangular matrix times its transposed complex conjugate), because it is uniquely defined and it automatically generates invertible matrices.*

(ii) *Moreover, regarding just* (2.14) *and* (2.15)*, it would even work if one uses different decompositions in* (4.1) *and* (4.2)*. This would lead to the same results in Section 5 except for Corollary 5.4, which will not remain valid, anymore.*

(iii) *In definition* (4.2)*, we follow the advice of Kreiß and Paparoditis (2003) and define* $\widehat{\mathbf{G}}(\omega)$ *via a nonparametric pre-whitening estimate of* $\mathbf{f}(\omega)$. *Asymptotically, we get the same results if we just set*

$$\widetilde{\mathbf{G}}(\omega)\overline{\widetilde{\mathbf{G}}(\omega)}^T = \frac{1}{n}\sum_{k=-N}^{N} K_h(\omega-\omega_k)\mathbf{I}_n(\omega_k)$$

and redefine $\widetilde{\mathbf{Q}}(\omega) = \widetilde{\mathbf{G}}(\omega)\widehat{B}(\omega)^{-1}$*, but for small sample sizes we expect slightly better results using* $\widehat{\mathbf{G}}(\omega)$. *Note, in the univariate case,* $\widetilde{\mathbf{Q}}(\omega)$ *agrees with* $\widetilde{q}(\omega)$ *as defined previous to* (2.9).

(iv) *Assumption* $(A1)$ *guarantees the positive definiteness of* $\mathbf{f}$ *and, for this reason, the pre-whitening estimate in* (4.2) *satisfies this property asymptotically (in probability). However, for very small sample sizes* n *relative to the dimension* r*, it may happen that the quantities on the right-hand sides of* (4.1) *and* (4.2) *are just positive semidefinite and not positive definite, which in turn disallows computation of their Cholesky decompositions. For medium and large sample sizes this problem practically does not occur. Hence, it is advisable to define*

$$\widetilde{\mathbf{Q}}(\omega) = \begin{cases} \widehat{\mathbf{G}}(\omega)\widehat{\mathbf{B}}(\omega)^{-1}, & \widehat{\mathbf{G}}(\omega) \text{ and } \widehat{\mathbf{B}}(\omega) \text{ exist} \\ \mathbf{I}_r, & \text{otherwise} \end{cases}$$

to overcome this difficulty of well-definition. Observe that in the second case, the hybrid bootstrap becomes the usual residual AR-bootstrap.

(v) *To obtain* $\widetilde{\mathbf{Q}}(\omega)$ *that satisfies* (2.14) *in its limit (in probability), it is essential to estimate* $\mathbf{f}_{AR}(\omega)$ *and* $\mathbf{f}(\omega)$ *separately and decompose them first, before defining* $\widetilde{\mathbf{Q}}(\omega)$ *as its product.*

Remark 4.2 (The special case $p = 0$)**.**
Setting $p = 0$*, this means that we do not fit any autoregressive model to the data* $\underline{X}_1, \ldots, \underline{X}_n$ *at all in Step 1 of our proposal. Actually, Step 2 shrivels to the standard i.i.d. bootstrap scheme obtaining* $\underline{X}_1^+, \ldots, \underline{X}_n^+$. *Although this ignores completely the dependency structure in* $\underline{X}_1, \ldots, \underline{X}_n$*, nevertheless, the hybrid bootstrap remains valid as discusssed in Section 5. In comparison, the nonparametric residual-based periodogram bootstrap (NPB) proposed by Franke and Härdle (1992) uses that the periodogram ordinates are asymptotically independently distributed according to an exponential distribution. For this reason, they resample in the frequency domain to obtain i.i.d. exponentially distributed random variates. In the case* $p = 0$*, in contrast, we do i.i.d. resampling in the time domain disregarding the dependency in the data and switch to the frequency domain afterwards by computing the discrete fourier transform. Observe that periodogram ordinates are just asymptotically independent, but for finite* n *this is not true anymore. Although we ignore the dependency contained in* $\underline{X}_1, \ldots, \underline{X}_n$ *by using this i.i.d. scheme setting* $p = 0$*, in comparison to the NPB, we get correlated periodogram ordinates in the frequency domain.*

5. Asymptotic theory and validity

This section is organized in three subsections. In the first one, we state the validity of our procedure for the multivariate sample mean, which constitutes an extension of the AAPB introduced by Kreiß and Paparoditis (2003), also in the univariate case. Validity for kernel spectral density matrix estimation and related quantities is discussed in the second subsection and, finally, the third deals with the asymptotic covariance structure of (entries of) empirical autocovariance matrices, their weak convergence in general as well as validity in some special situations. In the following, we use repeatedly Mallows' d_2-metric [cf. Mallows (1972)]. The d_2-distance between distributions $\mathcal{P}_1$ and $\mathcal{P}_2$ is defined as follows:

$$d_2\{\mathcal{P}_1, \mathcal{P}_2\} = \inf\{E|Y_1 - Y_2|^2\}^{1/2},$$

where the infimum is taken over all joint distributions for the pair of random variables Y_1 and Y_2 whose fixed marginal distributions are $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Compare Bickel and Freedman (1981) for a detailed discussion and related results.

5.1. **Sample mean.**

Theorem 5.1 (Validity for the sample mean)**.**
Suppose the assumptions (A1), (A2), (K1) and (B1) are satisfied. Then for all fixed $p \in \mathbb{N}_0$, it holds

$$d_2\{\mathcal{L}(\sqrt{n}\ \overline{\underline{X}}), \mathcal{L}(\sqrt{n}\ \overline{\underline{X}}^* | \underline{X}_1, \ldots, \underline{X}_n)\} \to 0$$

in probability, where $\overline{\underline{X}} = \frac{1}{n}\sum_{t=1}^n \underline{X}_t$ and $\overline{\underline{X}}^* = \frac{1}{n}\sum_{t=1}^n \underline{X}_t^*$.

5.2. **Spectral density estimates.**

Theorem 5.2 (Validity for spectral density estimates)**.**
Suppose the assumptions (A1), (A2), (A3), (K1), (K2), (K3) and (B3) are satisfied as well as $nb^5 \to C^2$ with a constant $C \geq 0$. Then for all fixed orders $p \in \mathbb{N}_0$ of the autoregressive fit, all $s \in \mathbb{N}$ and arbitrary frequencies $\omega_1, \ldots, \omega_s$ (not necessarily fourier frequencies), it holds

$$\begin{aligned} d_2\{&\mathcal{L}(\sqrt{nb}(\widehat{f}_{jk}(\omega_l) - f_{jk}(\omega_l)) : j,k = 1,\ldots,r; l = 1,\ldots,s), \\ &\mathcal{L}(\sqrt{nb}(\widehat{f}^*_{jk}(\omega_l) - \widetilde{f}_{jk}(\omega_l)) | \underline{X}_1, \ldots, \underline{X}_n : j,k = 1,\ldots,r; l = 1,\ldots,s)\} \to 0 \end{aligned}$$

in probability, where $\widehat{\mathbf{f}}(\omega) = \frac{1}{n}\sum_{j=-N}^N K_b(\omega - \omega_j)\mathbf{I}_n(\omega_j)$, $\widehat{\mathbf{f}}^*(\omega) = \frac{1}{n}\sum_{j=-N}^N K_b(\omega - \omega_j)\mathbf{I}^*_n(\omega_j)$ and $\widetilde{\mathbf{f}}(\omega) = \widetilde{\mathbf{Q}}(\omega)\widehat{\mathbf{f}}_{AR}(\omega)\overline{\widetilde{\mathbf{Q}}(\omega)}^T$.

A direct consequence of the above Theorem 5.2 is the corresponding result for the so-called cospectrum and quadrature spectrum, which are real-valued quantities and for this reason sometimes preferred to the complex-valued cross-spectral densities.

Corollary 5.1 (Cospectrum and quadrature spectrum)**.**
Putting $\mathbf{f}(\omega) = \frac{1}{2}(\mathbf{c}_{spec}(\omega) - i\mathbf{q}_{spec}(\omega))$ (analogue for $\widehat{\mathbf{f}}(\omega)$, $\widehat{\mathbf{f}}^*(\omega)$ and $\widetilde{\mathbf{f}}(\omega)$), we call the (real) matrix-valued quantities $\mathbf{c}_{spec}(\omega)$ and $\mathbf{q}_{spec}(\omega)$ the co- and quadrature spectral density matrices. Under the assumptions of Theorem 5.2 the following holds:

$$\begin{aligned} d_2\{&\mathcal{L}(\sqrt{nb}(\widehat{c}_{spec,jk}(\omega_l) - c_{spec,jk}(\omega_l)) : j,k = 1,\ldots,r; l = 1,\ldots,s), \\ &\mathcal{L}(\sqrt{nb}(\widehat{c}^*_{spec,jk}(\omega_l) - \widetilde{c}_{spec,jk}(\omega_l)) | \underline{X}_1, \ldots, \underline{X}_n : j,k = 1,\ldots,r; l = 1,\ldots,s)\} \to 0, \\ d_2\{&\mathcal{L}(\sqrt{nb}(\widehat{q}_{spec,jk}(\omega_l) - q_{spec,jk}(\omega_l)) : j,k = 1,\ldots,r; l = 1,\ldots,s), \\ &\mathcal{L}(\sqrt{nb}(\widehat{q}^*_{spec,jk}(\omega_l) - \widetilde{q}_{spec,jk}(\omega_l)) | \underline{X}_1, \ldots, \underline{X}_n : j,k = 1,\ldots,r; l = 1,\ldots,s)\} \to 0 \end{aligned}$$

in probability, respectively.

5.3. **Empirical autocovariances.**

Autocovariances provide a lot of information about the stochastic dependancy properties of a multivariate time series $\underline{\mathbf{X}}$. For instance, if one is interested in construction of confidence intervals, especially in the multivariate case, it is difficult to use existing central limit theorems to derive confidence regions. This is up to the sophisticated covariance matrix of the asymptotic normal distribution. Defining

$$\widehat{\mathbf{\Gamma}}(h) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-h}(\underline{X}_{t+h} - \overline{\underline{X}})(\underline{X}_t - \overline{\underline{X}})^T, & h \geq 0 \\ \frac{1}{n}\sum_{t=1-h}^{n}(\underline{X}_{t+h} - \overline{\underline{X}})(\underline{X}_t - \overline{\underline{X}})^T, & h < 0 \end{cases}, \tag{5.1}$$

namely, it holds [compare Hannan (1970), chapter IV, Section 3 and Theorem 14, p. 228]

$$\begin{aligned} & nCov(\widehat{\gamma}_{jk}(g) - \gamma_{jk}(g), \widehat{\gamma}_{lm}(h) - \gamma_{lm}(h)) \\ \to & \sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{r} \left(\sum_{\nu_1=-\infty}^{\infty} C_{\nu_1,j\mu_1} C_{\nu_1-g,k\mu_2} \right) \kappa_4(\mu_1,\mu_2,\mu_3,\mu_4) \left(\sum_{\nu_2=-\infty}^{\infty} C_{\nu_2,l\mu_3} C_{\nu_2-h,m\mu_4} \right) \\ & + \sum_{t=-\infty}^{\infty} \gamma_{km}(t)\gamma_{jl}(t-h+g) + \sum_{t=-\infty}^{\infty} \gamma_{kl}(t-h)\gamma_{jm}(t+g) \end{aligned} \tag{5.2}$$

for all $j,k,l,m = 1,\ldots,r$ and all lags $g,h \in \mathbb{Z}$, where $\kappa_4(\mu_1,\mu_2,\mu_3,\mu_4)$ is the fourth order joint cumulant between $\epsilon_{t,\mu_1}, \epsilon_{t,\mu_2}, \epsilon_{t,\mu_3}$ and ϵ_{t,μ_4}.

The first sum above containing these cumulants is difficult to handle and to interpret. For this reason, bootstrap methods can possibly help to overcome this difficulty. Desirable is to have a bootstrap procedure that is able to replicate the covariance structure in (5.2) as far as possible.

In the following two Theorems 5.3 and 5.4, we state the asymptotics for the hybrid bootstrap corresponding to (5.2) on the bootstrap level.

Theorem 5.3 (Asymptotic covariance structure)**.**
Assume (A1), (A2), (K1) and (B2) and let $p \in \mathbb{N}_0$. Defining $\mathbf{C}_p(\omega) = \frac{1}{\sqrt{2\pi}}\sum_{\nu=0}^{\infty}\mathbf{C}_\nu(p)e^{-i\nu\omega}$, where $\mathbf{C}_\nu(p)$, $\nu \in \mathbb{N}_0$ are the coefficient matrices of the causal representation of the best autoregressive fit of order p to the data in L_2-distance, for all $j,k,l,m = 1,\ldots,r$ and all $g,h \in \mathbb{Z}$, the following convergence in probability holds true:

$$\begin{aligned} & nCov^+(\widehat{\gamma}^*_{jk}(g) - E^+[\widehat{\gamma}^*_{jk}(g)], \widehat{\gamma}^*_{lm}(h) - E^+[\widehat{\gamma}^*_{lm}(h)]) \\ \to & \sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{r} \left(\int_{-\pi}^{\pi} (\mathbf{Q}(\omega_1)\mathbf{C}_p(\omega_1))_{j\mu_1} \left(\overline{\mathbf{C}_p(\omega_1)}^T \overline{\mathbf{Q}(\omega_1)}^T \right)_{\mu_2 k} e^{ig\omega_1} d\omega_1 \right) \\ & \kappa_4(p;\mu_1,\mu_2,\mu_3,\mu_4) \\ & \left(\int_{-\pi}^{\pi} (\mathbf{Q}(\omega_2)\mathbf{C}_p(\omega_2))_{l\mu_3} \left(\overline{\mathbf{C}_p(\omega_2)}^T \overline{\mathbf{Q}(\omega_2)}^T \right)_{\mu_4 m} e^{ih\omega_2} d\omega_2 \right) \\ & + \sum_{t=-\infty}^{\infty} \gamma_{km}(t)\gamma_{jl}(t-h+g) + \sum_{t=-\infty}^{\infty} \gamma_{kl}(t-h)\gamma_{jm}(t+g), \end{aligned} \tag{5.3}$$

where Cov^+ is the conditional covariance given $\underline{X}_1,\ldots,\underline{X}_n$, $\widehat{\mathbf{\Gamma}}^*(h)$ is the bootstrap analogue of (5.1) and $\kappa_4(p;\mu_1,\mu_2,\mu_3,\mu_4)$ is the fourth order joint cumulant between the corresponding components of the residuals obtained by the best autoregressive fit.

Theorem 5.4 (Asymptotic normality)**.**
Suppose the assumptions (A1), (A2), (A3), (K1), (K3) and (B3) are satisfied. Then for all

fixed $p \in \mathbb{N}_0$, all $s \in \mathbb{N}_0$ and lags $l = 0, \dots, s$, it holds

$$\mathcal{L}(\sqrt{n}(\widehat{\gamma}^*_{jk}(l) - E^+[\widehat{\gamma}^*_{jk}(l)])|\underline{X}_1, \dots, \underline{X}_n : j, k = 1, \dots, r; l = 0, \dots, s) \Rightarrow \mathcal{N}(\underline{0}, \mathbf{V})$$

in probability. Here, the asymptotic covariance matrix $\mathbf{V}$ can be constructed by the results of Theorem 5.3.

Unfortunately, the multivariate hybrid bootstrap method does not work completely satisfactory in general for autocovariances. In comparison to the AAPB this is not surprising, because in the univariate case the AAPB mimics the asymptotic distribution just for autocorrelations (ratio statistics) and not for autocovariances (spectral means), where the arising fourth order cumulant of the underlying white noise process is not captured properly (compare Theorem 4.1 (ii) and Corollary 4.1 (ii) in Kreiß and Paparoditis (2003)). The following corollary shows that our bootstrap procedure provides the same results as the AAPB in the univariate case.

Corollary 5.2 (Univariate case)**.**
Let $r = 1$. Under the assumptions of Theorem 5.4 we get

$$\mathcal{L}(\sqrt{n}(\widehat{\gamma}^*(l) - E^+[\widehat{\gamma}^*(l)])|X_1, \dots, X_n : l = 0, \dots, s) \Rightarrow \mathcal{N}(\underline{0}, V),$$

where V is obtained by

$$nCov^+(\widehat{\gamma}(g), \widehat{\gamma}(h)) \to \gamma(g)\gamma(h)(\eta(p) - 3) + \sum_{t=-\infty}^{\infty} \gamma(t)\gamma(t - h + g) + \sum_{t=-\infty}^{\infty} \gamma(t - h)\gamma(t + g) \quad (5.4)$$

in probability, where $E[(X_p - \sum_{\nu=1}^p a_\nu(p)X_{p-\nu})^2] = \sigma^2(p)$, $E[(X_p - \sum_{\nu=1}^p a_\nu(p)X_{p-\nu})^4] = \eta(p)\sigma^4(p)$ and $a_\nu(p)$, $\nu = 1, \dots, p$ are the coefficients of the best autoregressive fit of order p in L_2-distance.

Comparing the asymptotic covariances in (5.3) and (5.4) one striking difference regarding the first summands becomes obvious. The asymptotic covariance in the univariate case discussed in Corollary 5.2 depends exclusively through $\eta(p)$ on the initially fitted autoregressive model and therefore on the underlying hybrid bootstrap proposal. In contrast, the complicated covariance structure derived in Theorem 5.3 depends on the fourth order joint cumulants $\kappa_4(p; \mu_1, \mu_2, \mu_3, \mu_4)$ and, additionally, on the limit of $\widetilde{\mathbf{Q}}(\omega)$ as well as on the transfer function $\mathbf{C}_p(\omega)$ of the best autoregressive fit. Observe that all these aforementioned quantities depend on the order p of the autoregressive fit, which in turn causes the hybrid bootstrap as well as the AAPB to be not valid in this general setting. However, compared to the usual residual *AR*-bootstrap, the hybrid bootstrap is at least able to mimic exactly the second and third term in (5.2). In the upcoming Corollaries 5.3 and 5.4, two important special cases are presented where the hybrid bootstrap asymptotically works.

Apparently, both methods (AAPB and hybrid bootstrap) do not have the ability to imitate the fourth moments and accordingly the fourth order cumulants of the underlying white noise process $(\underline{\epsilon}_t : t \in \mathbb{Z})$ properly. This problem does not arise if we assume a normal distribution for the error process, because in this case all occurring fourth order cumulants vanish and we immediately obtain the following corollary.

Corollary 5.3 (Gaussian case)**.**
Assume that the residuals $(\underline{\epsilon}_t : t \in \mathbb{Z})$ are multivariate normally distributed. Under the assumptions of Theorem 5.4 for all $s \in \mathbb{N}_0$ and lags $l = 0, \dots, s$, it holds

$$\begin{aligned} d_2\{&\mathcal{L}(\sqrt{n}(\widehat{\gamma}_{jk}(l) - E[\widehat{\gamma}_{jk}(l)]) : j, k = 1, \dots, r; l = 0, \dots, s), \\ &\mathcal{L}(\sqrt{n}(\widehat{\gamma}^*_{jk}(l) - E^+[\widehat{\gamma}^*_{jk}(l)])|\underline{X}_1, \dots, \underline{X}_n : j, k = 1, \dots, r; l = 0, \dots, s)\} \to 0 \end{aligned}$$

in probability.

Assuming the underlying process $\underline{\mathbf{X}}$ to be a causal vector autoregressive time series of finite order $p_0 \in \mathbb{N}_0$ is another very important case. In this situation the usual residual bootstrap works well if we fit a model of order $p \geq p_0$. For this reason, we do not want the correction function $\widetilde{\mathbf{Q}}(\omega)$ to adjust anything and expect the hybrid bootstrap to be valid particularly in this case. Otherwise, this would represent a significant drawback compared to the residual bootstrap. The forthcoming corollary reinforces our speculation.

Corollary 5.4 ($VAR(p_0)$ case)**.**
Assume that the underlying observations $\underline{X}_1, \ldots, \underline{X}_n$ originate from a causal $VAR(p_0)$ model with $p_0 \in \mathbb{N}_0$, that is, the stationary process $\underline{\mathbf{X}}$ satisfies

$$\underline{X}_t = \sum_{k=1}^{p_0} \mathbf{A}_k \underline{X}_{t-k} + \underline{\epsilon}_t, \quad t \in \mathbb{Z}.$$

Under the assumptions of Theorem 5.4 for all $p \in \mathbb{N}_0$, $p \geq p_0$, all $s \in \mathbb{N}_0$ and lags $l = 0, \ldots, s$, it holds

$$d_2\{\mathcal{L}(\sqrt{n}(\widehat{\gamma}_{jk}(l) - E[\widehat{\gamma}_{jk}(l)]) : j,k = 1,\ldots,r; l = 0,\ldots,s), \\ \mathcal{L}(\sqrt{n}(\widehat{\gamma}^*_{jk}(l) - E^+[\widehat{\gamma}^*_{jk}(l)])|\underline{X}_1,\ldots,\underline{X}_n : j,k = 1,\ldots,r; l = 0,\ldots,s)\} \to 0$$

in probability.

6. Proofs and auxiliary results

6.1. **The nonparametric correction function.**

Lemma 6.1 (Consistency of the correction function)**.**
Assume (A1), (A2), (K1) and (B1). Then, for the nonparametric correction function $\widetilde{\mathbf{Q}}(\omega) = \widehat{\mathbf{G}}(\omega)\widehat{\mathbf{B}}(\omega)^{-1}$ as defined in (4.1) *and* (4.2) *(note the suppressed dependency on the sample size n), it holds*

$$\widetilde{\mathbf{Q}}(\omega) \to \mathbf{Q}(\omega)$$

in probability for all ω, where $\mathbf{Q}(\omega) = \mathbf{G}(\omega)\mathbf{B}(\omega)^{-1}$ with Cholesky decompositions $\mathbf{B}(\omega)\overline{\mathbf{B}(\omega)}^T = \mathbf{f}_{AR}(\omega)$ and $\mathbf{G}(\omega)\overline{\mathbf{G}(\omega)}^T = \mathbf{f}(\omega)$. If even (B2) is satisfied, we get the uniform convergence

$$\sup_{\omega} \|\widetilde{\mathbf{Q}}(\omega) - \mathbf{Q}(\omega)\| = o_P(1)$$

and if additionally (A3), (K3) and (B3) are fulfilled, the first three (entrywise) derivatives of $\widetilde{\mathbf{Q}}(\omega)$ exists and we get the uniform convergence in probability of the first two, that is

$$\sup_{\omega} \|\widetilde{\mathbf{Q}}^{(j)}(\omega) - \mathbf{Q}^{(j)}(\omega)\| = o_P(1), \quad j = 1,2$$

and the boundedness in probability of the third, that is $\sup_{\omega} \|\widetilde{\mathbf{Q}}^{(3)}(\omega)\| = O_P(1)$.

Proof.
First of all, we discuss some preliminary considerations. The Cholesky decomposition $\mathbf{B}\overline{\mathbf{B}}^T = \mathbf{A}$ of a (complex) positive definite matrix $\mathbf{A}$ is obtained recursively by

$$b_{kl} = \begin{cases} 0, & k < l \\ (a_{kk} - \sum_{j=1}^{k-1} b_{kj}\overline{b_{kj}})^{1/2}, & k = l \\ \frac{1}{b_{ll}}(a_{kl} - \sum_{j=1}^{l-1} b_{kj}\overline{b_{lj}}), & k > l \end{cases}, \tag{6.1}$$

where $\mathbf{B}$ is uniquely defined and all diagonal elements are real-valued and strictly positive and therefore $\mathbf{B}$ is invertible. Assuming a matrix-valued function $\mathbf{A}(\omega)$ to be positive definite for all ω, the same properties hold for its Cholesky decomposition $\mathbf{B}(\omega)$. Further, if we assume $\mathbf{A}(\omega)$ to be (entrywise) k-times differentiable in ω, this property is also satisfied for $\mathbf{B}(\omega)$, which can be seen easily computing the derivatives according to (6.1). Moreover, if $(\mathbf{A}_n(\omega) : n \in \mathbb{N})$ is a sequence of matrix-valued functions assumed to be positive definite as well as k-times (entrywise) differentiable for all ω, uniform convergence of their first k derivatives $\mathbf{A}_n^{(d)}(\omega)$, $d = 0, 1, \ldots, k$ causes uniform convergence of the k-th derivative $\mathbf{B}_n^{(k)}(\omega)$ of the corresponding Cholesky decomposition $\mathbf{B}_n(\omega)$.

Since the spectral densities $\mathbf{f}(\omega)$ and $\mathbf{f}_{AR}(\omega)$ are both positive definite for all ω due to the assumptions (A1) and (A2) and because the Yule-Walker estimates always yield to stable autoregressive models [compare Whittle (1963)], it suffices to restrict considerations to the convergence of the quantities on the right-hand sides of (4.1) and (4.2) to $\mathbf{f}(\omega)$ and $\mathbf{f}_{AR}(\omega)$ respectively as well as the convergence of their derivatives. We prove only the most sophisticated assertion for $\widetilde{\mathbf{Q}}^{(2)}(\omega)$.

The uniform convergence of $\widehat{\mathbf{f}}_{AR}(\omega)$ in probability follows by standard arguments using (6.7) below and, because of the positive definiteness of its limit $\mathbf{f}_{AR}(\omega)$, we can treat $\widehat{\mathbf{f}}_{AR}(\omega)$ as a positive definite matrix for sufficiently large n (in probability). Hence, the right-hand side in (4.2) is well defined for large n (in probability). Entrywise geometrically decaying coefficient matrices of the causal representation of the (stable) autoregressive fit yield uniform convergence for all derivatives of $\widehat{\mathbf{f}}_{AR}(\omega)$ and the same holds true for its inverse $\widehat{\mathbf{f}}_{AR}^{-1}(\omega)$, causing the k-th derivatives of $\widehat{\mathbf{B}}(\omega)$ and $\widehat{\mathbf{B}}^{-1}(\omega)$ to converge uniformly, also. Now, consider the term on the right-hand side of (4.2) more closely and define

$$\widehat{\mathbf{Q}}(\omega) = \frac{1}{n}\sum_{k=-N}^{N} K_h(\omega-\omega_k)\widehat{\mathbf{B}}(\omega_k)^{-1}\mathbf{I}_n(\omega_k)\overline{\widehat{\mathbf{B}}(\omega_k)^{-1}}^T.$$

Thanks to the uniform convergence of $\widehat{\mathbf{B}}(\omega)$ and $\widehat{\mathbf{B}}^{-1}(\omega)$ and their derivatives, it remains to show

$$\sup_{\omega}\|\widehat{\mathbf{Q}}^{(d)}(\omega) - (\mathbf{B}^{-1}(\omega)\mathbf{f}(\omega)\overline{\mathbf{B}^{-1}(\omega)}^T)^{(d)}\| = o_P(1).$$

A Taylor series expansion yields

$$\begin{aligned}&\widehat{\mathbf{Q}}^{(d)}(\omega)\\ =\ &\sum_{s_1,s_2=0}^{d}(\widehat{\mathbf{B}}^{-1}(\omega))^{(s_1)}\left(\frac{1}{nh^{d+1}}\sum_{k=-N}^{N}K^{(d)}\left(\frac{\omega-\omega_k}{h}\right)(\omega_k-\omega)^{s_1+s_2}\mathbf{I}_n(\omega_k)\right)\overline{(\widehat{\mathbf{B}}^{-1}(\omega))^{(s_2)}}^T\\ &+O_P(h)\end{aligned} \tag{6.2}$$

uniformly in ω and it remains to check the following uniform convergence for the expression in the big round parentheses in (6.2):

$$\sup_{\omega}\|\frac{1}{nh^{d+1}}\sum_{k=-N}^{N}K^{(d)}\left(\frac{\omega-\omega_k}{h}\right)(\omega_k-\omega)^s\mathbf{I}_n(\omega_k) - \frac{d!}{(d-s)!}\mathbf{f}^{(d-s)}(\omega)\| = o_P(1)$$

for $d = 0, 1, 2$ and $s = 0, 1, \ldots, d$. Observe that all sums in (6.2) with $s = s_1 + s_2 > d$ can be neglected because they vanish asymptotically with $O_P(h^{s-d})$ due to assumption (K3). To prove the last assertion, we follow the idea of Franke and Härdle (1992, Theorem A1). Initially, the last supremum is bounded by

$$\sup_\omega \| \frac{1}{nh^{d+1}} \sum_{k=-N}^{N} K^{(d)}\left(\frac{\omega-\omega_k}{h}\right)(\omega_k-\omega)^s(\mathbf{I}_n(\omega_k)-\mathbf{C}(\omega_k)\mathbf{I}_{n,\epsilon}(\omega_k)\overline{\mathbf{C}(\omega_k)}^T)\| \quad (6.3)$$

$$+ \sup_\omega \| \frac{1}{nh^{d+1}} \sum_{k=-N}^{N} K^{(d)}\left(\frac{\omega-\omega_k}{h}\right)(\omega_k-\omega)^s\mathbf{C}(\omega_k)\mathbf{I}_{n,\epsilon}(\omega_k)\overline{\mathbf{C}(\omega_k)}^T$$

$$-\frac{1}{(d-s)!}\mathbf{f}^{(d-s)}(\omega)\frac{1}{nh}\sum_{k=-N}^{N} K^{(d)}(\frac{\omega-\omega_k}{h})(\frac{\omega_k-\omega}{h})^d\| \quad (6.4)$$

$$+ \sup_\omega \| \frac{1}{(d-s)!}\mathbf{f}^{(d-s)}(\omega)\frac{1}{nh}\sum_{k=-N}^{N} K^{(d)}(\frac{\omega-\omega_k}{h})(\frac{\omega_k-\omega}{h})^d - \frac{d!}{(d-s)!}\mathbf{f}^{(d-s)}(\omega)\|, \quad (6.5)$$

where $\mathbf{C}(\omega) := \sum_{\nu=-\infty}^{\infty}\mathbf{C}_\nu e^{-i\nu\omega}$ and $\mathbf{I}_{n,\epsilon}(\omega)$ is the periodogram based on $\underline{\epsilon}_1,\ldots,\underline{\epsilon}_n$. Now, we consider these three expressions seperately.

Theorem 2 in Hannan (1970, p.248) indicates $\|\mathbf{I}_n(\omega)-\mathbf{C}(\omega)\mathbf{I}_{n,\epsilon}(\omega)\overline{\mathbf{C}(\omega)}^T\| = O_P(n^{-1/2})$ uniformly in ω and the supremum in (6.3) and in (6.5) vanish asymptotically in probability by standard arguments. Using again Taylor expansion for $\mathbf{C}(\omega_k)$ the supremum in (6.4) can be bounded by

$$\sup_\omega \| \sum_{j_1,j_2=0}^{d} \mathbf{C}^{(j_1)}(\omega)\left(\frac{1}{nh^{d+1}}\sum_{k=-N}^{N} K^{(d)}\left(\frac{\omega-\omega_k}{h}\right)\frac{(\omega_k-\omega)^{s+j_1+j_2}}{j_1!j_2!}\mathbf{I}_{n,\epsilon}(\omega_k)\right)\overline{\mathbf{C}^{(j_2)}(\omega)}^T$$

$$-\frac{(-1)^d}{(d-s)!}\mathbf{f}^{(d-s)}(\omega)\frac{1}{nh}\sum_{k=-N}^{N} K^{(d)}(\frac{\omega-\omega_k}{h})(\frac{\omega-\omega_k}{h})^d\| + O_P(h).$$

Now, for instance, a multivariate version of Theorem 5.9.1 in Brillinger (1981) and following the approach of Franke and Härdle (1992) for proving Theorem A1 yield the claimed uniform convergence in probability of $\widetilde{\mathbf{Q}}(\omega)$ as n tends to infinity. Here, $(nh^6)^{-1} = O(1)$ has to be satisfied in comparison to Franke and Härdle, where no derivatives are estimated. □

6.2. **Sample mean.**

Proof of Theorem 5.1.
Since convergence in d_2 metric is equivalent to weak convergence and convergence of the first two moments [compare Bickel and Freedman (1981), Lemma 8.3], it suffices to show

$$Var^+(\sqrt{n}\ \underline{\overline{X}}^*) \to 2\pi\mathbf{f}(0),$$

where Var^+ is the conditional variance given $\underline{X}_1,\ldots,\underline{X}_n$ and

$$\mathcal{L}\{\sqrt{n}\ \underline{\overline{X}}^*|\underline{X}_1,\ldots,\underline{X}_n\} \Rightarrow \mathcal{N}(0,2\pi\mathbf{f}(0))$$

in probability, respectively. Recall that $Var(\sqrt{n}\underline{X}) \to 2\pi\mathbf{f}(0)$ and $\sqrt{n}\underline{X} \Rightarrow \mathcal{N}(0, 2\pi\mathbf{f}(0))$ as $n \to \infty$ [compare Brockwell and Davis (1991), p.406]. Straightforward calculation yields

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\underline{X}_t^* &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sqrt{\frac{2\pi}{n}}\sum_{j=-[n/2]}^{[n/2]}\widetilde{\mathbf{Q}}(\omega_j)\underline{J}_n^+(\omega_j)e^{it\omega_j} \\
&= \frac{\sqrt{2\pi}}{n}\sum_{j=-[n/2]}^{[n/2]}\widetilde{\mathbf{Q}}(\omega_j)\underline{J}_n^+(\omega_j)\sum_{t=1}^{n}e^{it\omega_j} \\
&= \sqrt{2\pi}\widetilde{\mathbf{Q}}(0)\underline{J}_n^+(0) \\
&= \widetilde{\mathbf{Q}}(0)\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\underline{X}_t^+\right)
\end{aligned}$$

and for the covariance matrix, we get immediately

$$Var^+\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\underline{X}_t^*\right) = \widetilde{\mathbf{Q}}(0)Var^+\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\underline{X}_t^+\right)\widetilde{\mathbf{Q}}(0)^T.$$

For this reason, the claimed convergence in Mallows' metric follows from

$$\mathcal{L}\{\sqrt{n}\ \overline{\underline{X}}^+|\underline{X}_1,\ldots,\underline{X}_n\} \Rightarrow \mathcal{N}(0, 2\pi\mathbf{f}_{AR}(0)) \tag{6.6}$$

in probability, because, by construction, $2\pi\mathbf{Q}(0)\mathbf{f}_{AR}(0)\mathbf{Q}(0)^T = 2\pi\mathbf{f}(0)$. Using the Cramer-Wold device, assertion (6.6) results from an adequate CLT, e.g. for weak dependent random variables as derived by Neumann and Paparoditis (2008, Theorem 6.1), which is well suited for the bootstrap. Thereby, we employ the convergence rate

$$\sup_{\nu\in\mathbb{N}_0}\|\widehat{\mathbf{C}}_\nu(p) - \mathbf{C}_\nu(p)\| = \frac{1}{r^\nu}O_P(n^{1/2}), \tag{6.7}$$

for some $r > 1$ which was established by Kreiß (1984, p.7) for the coefficient matrices $\widehat{\mathbf{C}}_\nu(p)$, $\nu \in \mathbb{N}_0$ of the causal representation of the autoregressive fit of order p, using a multidimensional version of Cauchy's inequality for holomorphic functions [compare Kreiß and Franke (1992), Lemma 2.2 in the univariat case]. □

6.3. **Spectral density.**

Proof of Theorem 5.2.
To prove the Theorem, it is more convenient to use the *vec*-operator that creates a column vector stacking the columns of a matrix below one another and to show the sufficient assertion

$$\begin{aligned}
&d_2\{\mathcal{L}(\sqrt{nb}vec\left(\left[\widehat{\mathbf{f}}(\omega_1) - \mathbf{f}(\omega_1)|\cdots|\widehat{\mathbf{f}}(\omega_s) - \mathbf{f}(\omega_s)\right]\right)), \\
&\qquad\mathcal{L}(\sqrt{nb}vec\left(\left[\widehat{\mathbf{f}}^*(\omega_1) - \widetilde{\mathbf{f}}(\omega_1)|\cdots|\widehat{\mathbf{f}}^*(\omega_s) - \widetilde{\mathbf{f}}(\omega_s)\right]\right)|\underline{X}_1,\ldots,\underline{X}_n)\} \to 0
\end{aligned}$$

in probability. By Lemma 8.8 of Bickel and Freedman (1981), we can split the squared Mallows' metric in a variance part $V_n^2(\omega)$ and a squared bias part $b_n^2(\omega)$, where

$$\begin{aligned}
V_n^2(\omega) &= d_2\{\mathcal{L}(\sqrt{nb}vec\left(\left[\widehat{\mathbf{f}}(\omega_1) - E[\widehat{\mathbf{f}}(\omega_1)]|\cdots|\widehat{\mathbf{f}}(\omega_s) - E[\widehat{\mathbf{f}}(\omega_s)]\right]\right)), \\
&\qquad\mathcal{L}(\sqrt{nb}vec\left(\left[\widehat{\mathbf{f}}^*(\omega_1) - E^+[\widehat{\mathbf{f}}^*(\omega_1)]|\cdots|\widehat{\mathbf{f}}^*(\omega_s) - E^+[\widehat{\mathbf{f}}^*(\omega_s)]\right]\right)|\underline{X}_1,\ldots,\underline{X}_n)\}
\end{aligned}$$

and

$$b_n^2(\omega) = nb\|vec\left(\left[E[\widehat{\mathbf{f}}(\omega_1)] - \mathbf{f}(\omega_1)|\cdots|E[\widehat{\mathbf{f}}(\omega_s)] - \mathbf{f}(\omega_s)\right]\right) - vec\left(\left[E^+[\widehat{\mathbf{f}}^*(\omega_1)] - \widetilde{\mathbf{f}}(\omega_1)|\cdots|E^+[\widehat{\mathbf{f}}^*(\omega_s)] - \widetilde{\mathbf{f}}(\omega_s)\right]\right)\|^2$$

and by Lemma 8.3 of the same paper, convergence in the d_2 metric is equivalent to weak convergence and convergence of the first two moments. The latter two follows from Lemma 6.2 and the weak convergence is a consequence of Lemma 6.3, so that $V_n^2(\omega) = o_P(1)$ holds. Recall that

$$nbCov(\widehat{f}_{jk}(\omega), \widehat{f}_{lm}(\lambda)) \to \begin{cases} \{f_{jl}(\omega)f_{mk}(\omega) + f_{jm}(\omega)f_{kl}(\omega)\}\frac{1}{2\pi}\int K^2(u)du, & \omega = \lambda \in \{0,\pi\} \\ f_{jl}(\omega)f_{mk}(\omega)\frac{1}{2\pi}\int K^2(u)du, & 0 < \omega = \lambda < \pi \\ 0, & \omega \neq \lambda \end{cases} \quad (6.8)$$

and

$$\sqrt{nb}(\widehat{f}_{jk}(\omega_l) - E[\widehat{f}_{jk}(\omega_l)] : j,k = 1,\ldots,r; l = 1,\ldots,s)$$

is asymptotically (complex) normally distributed with mean vector $\underline{0}$ and covariance matrix obtained by (6.8) [compare Hannan (1970), Theorem 9, p. 280 and Theorem 11, p. 289 for a different but asymptotically equivalent estimator]. Note that assumption (5.2) in Hannan (1970) is avoided in this context. Finally, the required convergence of $b_n^2(\omega)$ results from

$$E[\widehat{\mathbf{f}}(\omega)] - \mathbf{f}(\omega) \to \frac{C}{4\pi}\mathbf{f}''(\omega)\int K(u)u^2du$$

for $nb^5 \to C^2 \geq 0$ as $n \to \infty$ and Lemma 6.4 below.

Lemma 6.2 (Covariance structure).
Assume (A1), (A2), (K1) and (B2). For $j,k,l,m \in \{1,\ldots,r\}$ and $\omega,\lambda \in [0,\pi]$, the following convergence in probability holds true:

$$nbCov^+(\widehat{f}^*_{jk}(\omega), \widehat{f}^*_{lm}(\lambda)) \to \begin{cases} \{f_{jl}(\omega)f_{mk}(\omega) + f_{jm}(\omega)f_{kl}(\omega)\}\frac{1}{2\pi}\int K^2(u)du, & \omega = \lambda \in \{0,\pi\} \\ f_{jl}(\omega)f_{mk}(\omega)\frac{1}{2\pi}\int K^2(u)du, & 0 < \omega = \lambda < \pi \\ 0, & \omega \neq \lambda \end{cases}$$

Proof.
Let $k_1, k_2, h_1, h_2 \in \{1,\ldots,r\}$ and $\omega,\lambda \in [0,\pi]$, then insertion and straightforward calculation yields

$$\begin{aligned} & nbCov^+(\widehat{f}^*_{k_1h_1}(\omega), \widehat{f}^*_{k_2h_2}(\lambda)) \\ = & \frac{b}{n}\sum_{j=-N}^{N}\sum_{l=-N}^{N}K_b(\omega-\omega_j)K_b(\omega-\omega_l)\sum_{m_1,m_2,m_3,m_4=1}^{r}\widetilde{q}_{k_1m_1}(\omega_j)\overline{\widetilde{q}_{h_1m_2}(\omega_j)}\widetilde{q}_{k_2m_3}(\omega_l)\overline{\widetilde{q}_{h_2m_4}(\omega_l)} \\ & (E^+[I^+_{n,m_1m_2}(\omega_j)I^+_{n,m_3m_4}(\omega_l)] - E^+[I^+_{n,m_1m_2}(\omega_j)]E^+[I^+_{n,m_3m_4}(\omega_l)]) \\ = & \frac{b}{n}\sum_{j=-N}^{N}\sum_{l=-N}^{N}K_b(\omega-\omega_j)K_b(\omega-\omega_l)\sum_{m_1,m_2,m_3,m_4=1}^{r}\widetilde{q}_{k_1m_1}(\omega_j)\overline{\widetilde{q}_{h_1m_2}(\omega_j)}\widetilde{q}_{k_2m_3}(\omega_l)\overline{\widetilde{q}_{h_2m_4}(\omega_l)} \\ & \frac{1}{4\pi^2n^2}\sum_{s,t,u,v=1}^{n}\sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty}\sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{r}\widehat{C}_{\nu_1,m_1\mu_1}(p)\widehat{C}_{\nu_2,m_2\mu_2}(p)\widehat{C}_{\nu_3,m_3\mu_3}(p)\widehat{C}_{\nu_4,m_4\mu_4}(p) \\ & (E^+[\epsilon^+_{s-\nu_1,\mu_1}\epsilon^+_{t-\nu_2,\mu_2}\epsilon^+_{u-\nu_3,\mu_3}\epsilon^+_{v-\nu_4,\mu_4}] \\ & \quad -E^+[\epsilon^+_{s-\nu_1,\mu_1}\epsilon^+_{t-\nu_2,\mu_2}]E^+[\epsilon^+_{u-\nu_3,\mu_3}\epsilon^+_{v-\nu_4,\mu_4}])e^{-i(s-t)\omega_j}e^{-i(u-v)\omega_l}. \end{aligned} \quad (6.9)$$

Here, for the first equality we used $\mathbf{I}_n^*(\omega) = \widetilde{Q}(\omega)\mathbf{I}_n^+(\omega)\overline{\widetilde{Q}(\omega)}^T$ and the second results from inserting for the periodogram. Because of the identity next to (6.17)-(6.19) we can deal with those three summands separately. Initially, we consider (6.18). Here, $\sum_{t=1}^n e^{it\omega} = 0$ if $\omega \neq 0$ and n otherwise causes the sum over j_2 in (6.16) to collapse and a rearrangement results in

$$\begin{aligned}
&\frac{b}{n}\sum_{j=-N}^{N} K_b^2(\omega-\omega_j) \\
&\sum_{m_1,m_4=1}^{r} \widetilde{q}_{k_1 m_1}(\omega_j) \sum_{\mu_1,\mu_4=1}^{r} \frac{1}{2\pi}\sum_{\nu_1=0}^{\infty} \widehat{C}_{\nu_1,m_1\mu_1} e^{-i\nu_1\omega_j} \widehat{\sigma}_{\mu_1\mu_4}(p) \overline{\sum_{\nu_4=0}^{\infty} \widehat{C}_{\nu_4,m_4\mu_4} e^{-i\nu_4\omega_j}}\, \overline{\widetilde{q}_{h_2 m_4}(\omega_j)} \\
&\sum_{m_3,m_2=1}^{r} \widetilde{q}_{k_2 m_3}(\omega_j) \sum_{\mu_3,\mu_2=1}^{r} \frac{1}{2\pi}\sum_{\nu_3=0}^{\infty} \widehat{C}_{\nu_3,m_3\mu_3} e^{-i\nu_3\omega_j} \widehat{\sigma}_{\mu_3\mu_2}(p) \overline{\sum_{\nu_2=0}^{\infty} \widehat{C}_{\nu_2,m_2\mu_2} e^{i\nu_2\omega_j}}\, \overline{\widetilde{q}_{h_1 m_2}(\omega_j)} \\
= \; &\frac{b}{n}\sum_{j=-N}^{N} K_b^2(\omega-\omega_j) \left(\widetilde{\mathbf{Q}}(\omega_j)\widehat{\mathbf{f}}_{\mathbf{AR}}(\omega_j)\overline{\widetilde{\mathbf{Q}}(\omega_j)}^T\right)_{k_1h_2} \left(\widetilde{\mathbf{Q}}(\omega_j)\widehat{\mathbf{f}}_{\mathbf{AR}}(\omega_j)\overline{\widetilde{\mathbf{Q}}(\omega_j)}^T\right)_{k_2h_1}.
\end{aligned}$$

Because of the uniform convergence in ω of the quantities $\widetilde{\mathbf{Q}}(\omega)$ and $\widehat{\mathbf{f}}_{AR}(\omega)$, the last sum is equal to

$$\begin{aligned}
&\frac{b}{n}\sum_{j=-N}^{N} K_b^2(\omega-\omega_j) \left(\mathbf{Q}(\omega_j)\mathbf{f}_{\mathbf{AR}}(\omega_j)\overline{\mathbf{Q}(\omega_j)}^T\right)_{k_1h_2} \left(\mathbf{Q}(\omega_j)\mathbf{f}_{\mathbf{AR}}(\omega_j)\overline{\mathbf{Q}(\omega_j)}^T\right)_{k_2h_1} + o_P(1) \\
= \; &\frac{1}{nb}\sum_{j=-N}^{N} K^2\left(\frac{\omega-\omega_j}{b}\right) f_{k_1h_2}(\omega_j) f_{k_2h_1}(\omega_j) + o_P(1),
\end{aligned}$$

where we used the correcting property of $\widetilde{\mathbf{Q}}(\omega)$. Concerning assumption $(A1)$, the spectral density $\mathbf{f}(\omega)$ is componentwise differentiable with bounded derivative. For this reason, Taylor expansions of $f_{k_1h_2}(\omega_j)$ and $f_{k_2h_1}(\omega_j)$ plus the converging Riemann sum yield

$$\begin{aligned}
&\frac{1}{nb}\sum_{j=-N}^{N} K^2\left(\frac{\omega-\omega_j}{b}\right) f_{k_1h_2}(\omega_j) f_{k_2h_1}(\omega_j) + o_P(1) \\
= \; & f_{k_1h_2}(\omega) f_{k_2h_1}(\omega) \frac{1}{2\pi}\left(\frac{2\pi}{nb}\sum_{j=-N}^{N} K^2\left(\frac{\omega-\omega_j}{b}\right)\right) + O_P(b) + o_P(1) \\
\to \; & f_{k_1h_2}(\omega) f_{k_2h_1}(\omega) \frac{1}{2\pi}\int_{-\pi}^{\pi} K^2(x)dx
\end{aligned}$$

in probability. Arguments are similar for the term related to (6.19) and one gets

$$f_{k_1h_2}(\omega) f_{k_2h_1}(\omega) \frac{1}{2\pi}\left(\frac{2\pi}{nb}\sum_{j=-N}^{N} K\left(\frac{\omega-\omega_j}{b}\right) K\left(\frac{\omega+\omega_j}{b}\right)\right) + O_P(b) + o_P(1),$$

where the involved Riemann sum converges to zero for $\omega \in (0,\pi)$ and to $\frac{1}{2\pi}\int K^2(u)du$ for $\omega \in \{0,\pi\}$ as required. It remains to check (6.17) concerning its asymptotic behaviour. Inserting (6.17) in equation (6.9) and standard calculations result in an $O_P(b) = o_P(1)$ term that, for this reason, does not play a role asymptotically. This completes the proof. □

Lemma 6.3 (Asymptotic normality).
Assume (A1), (A2), (K1), (K2) and (B2). Then, the following assertion holds true:

$$\mathcal{L}\left[\sqrt{nb}\text{vec}\left(\left[\widehat{\mathbf{f}}^*(\omega_1)-E^+[\widehat{\mathbf{f}}^*(\omega_1)]|\cdots|\widehat{\mathbf{f}}^*(\omega_s)-E^+[\widehat{\mathbf{f}}^*(\omega_s)]\right]\right)\Big|\,\underline{X}_1,\ldots,\underline{X}_n\right]\Rightarrow\mathcal{N}^{\mathbb{C}}(\underline{0},\mathbf{W})$$

in probability, where $\mathcal{N}^{\mathbb{C}}$ denotes a complex normal distribution [cf. Brillinger (1981), p.89] and the asymptotic covariance matrix $\mathbf{W}$ is obtained by the results of Lemma 6.2.

Proof.
Let $\underline{c}=(\underline{c}^{(1)},\ldots,\underline{c}^{(s)})^T\in\mathbb{C}^{sr^2}$ with $\underline{c}^{(l)}\in\mathbb{C}^{r^2}$, $l=1,\ldots,s$. Using the Cramer-Wold device, applied to complex-valued random variables, it suffices to show asymptotic normality for

$$\begin{aligned}&\underline{c}^T\sqrt{nb}vec\left(\left[\widehat{\mathbf{f}}^*(\omega_1)-E^+[\widehat{\mathbf{f}}^*(\omega_1)]|\cdots|\widehat{\mathbf{f}}^*(\omega_s)-E^+[\widehat{\mathbf{f}}^*(\omega_s)]\right]\right)\\ =\;&\sum_{l=1}^{s}\underline{c}^{(l)\,T}\sqrt{nb}vec\left(\left[\widehat{\mathbf{f}}^*(\omega_l)-E^+[\widehat{\mathbf{f}}^*(\omega_l)]\right]\right).\end{aligned}$$

For this reason, without loss of generality, we can restrict our considerations to the case $s=1$. Analogue to Theorem 2 in Hannan (1970, p.248), it holds

$$\mathbf{I}_n^+(\omega)=\left(\sum_{\nu=0}^{\infty}\widehat{\mathbf{C}}_\nu(p)e^{-i\nu\omega}\right)\widehat{\mathbf{L}}(p)\mathbf{I}_{n,\epsilon^+}(\omega)\overline{\widehat{\mathbf{L}}(p)}^T\overline{\left(\sum_{\nu=0}^{\infty}\widehat{\mathbf{C}}_\nu(p)e^{-i\nu\omega}\right)}^T+O_{P^*}(n^{-\frac{1}{2}}),$$

where $\widehat{\mathbf{L}}(p)$ is defined in Step 1 in Section 4 and $\mathbf{I}_{n,\epsilon^+}(\omega)$ is the periodogram based on the bootstrap residuals $\underline{\epsilon}_1^+,\ldots,\underline{\epsilon}_n^+$. Using this formula and $\mathbf{I}_n^*(\omega)=\widetilde{\mathbf{Q}}(\omega)\mathbf{I}_n^+(\omega)\overline{\widetilde{\mathbf{Q}}(\omega)}^T$, we get

$$\sqrt{nb}(\widehat{\mathbf{f}}^*(\omega)-E^+[\widehat{\mathbf{f}}^*(\omega)])=\sqrt{\frac{b}{n}}\sum_{j=-N}^{N}K_b(\omega-\omega_j)\widehat{\mathbf{M}}(\omega_j)\left(\mathbf{I}_{n,\epsilon^+}(\omega_j)-\frac{1}{2\pi}\mathbf{I}_r\right)\overline{\widehat{\mathbf{M}}(\omega_j)}^T+o_{P^*}(1),$$

where $\widehat{\mathbf{M}}(\omega)=\widetilde{\mathbf{Q}}(\omega)(\sum_{\nu=0}^{\infty}\widehat{\mathbf{C}}_\nu(p)e^{-i\nu\omega})\widehat{\mathbf{L}}(p)$. Thanks to a multivariate analogue to Theorem 5.9.1 in Brillinger (1981), instead of the first term on the right-hand side of the above equality, we can consider the asymptotically equivalent statistic

$$\begin{aligned}&\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi}K_b(\omega-x)\mathbf{M}(x)\left(\mathbf{I}_{n,\epsilon^+}(x)-\frac{1}{2\pi}\mathbf{I}_r\right)\overline{\mathbf{M}(x)}^Tdx\\ =\;&\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi}K(u)\mathbf{M}(\omega-ub)\left(\mathbf{I}_{n,\epsilon^+}(\omega-ub)-\frac{1}{2\pi}\mathbf{I}_r\right)\overline{\mathbf{M}(\omega-ub)}^Tdu \qquad (6.10)\\ =\;&\mathbf{M}(\omega)\left(\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi}K(u)\left(\mathbf{I}_{n,\epsilon^+}(\omega-ub)-\frac{1}{2\pi}\mathbf{I}_r\right)du\right)\overline{\mathbf{M}(\omega)}^T+\mathbf{D}_{n,1}^+(\omega)+\mathbf{D}_{n,2}^+(\omega),\end{aligned}$$

where $\mathbf{M}(\omega)=\mathbf{Q}(\omega)\left(\sum_{\nu=0}^{\infty}\mathbf{C}_\nu(p)e^{-i\nu\omega}\right)\mathbf{L}(p)$ is the limit in probability of $\widehat{\mathbf{M}}(\omega)$ and the quantaties $\mathbf{D}_{n,1}^+(\omega)$ and $\mathbf{D}_{n,2}^+(\omega)$ are defined as follows:

$$\begin{aligned}\mathbf{D}_{n,1}^+(\omega)&=\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi}K(u)(\mathbf{M}(\omega-ub)-\mathbf{M}(\omega))\left(\mathbf{I}_{n,\epsilon^+}(\omega-ub)-\frac{1}{2\pi}\mathbf{I}_r\right)\overline{\mathbf{M}(\omega-ub)}^Tdu,\\ \mathbf{D}_{n,2}^+(\omega)&=\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi}K(u)\mathbf{M}(\omega-ub)\left(\mathbf{I}_{n,\epsilon^+}(\omega-ub)-\frac{1}{2\pi}\mathbf{I}_r\right)\overline{(\mathbf{M}(\omega-ub)-\mathbf{M}(\omega))}^Tdu.\end{aligned}$$

For the components of $\mathbf{D}_{n,k}^+(\omega)$, $k\in\{1,2\}$, straightforward calculations yield $E^+[D_{n,k}^+(i,j)(\omega)]=0$ and

$$E^+[|D_{n,k}^+(i,j)(\omega)|^2]=O_P(\max_{i,j=1,\ldots,r}\{|M_{i,j}(\omega-ub)-M_{i,j}(\omega)|^2\})=O_P(b^2)$$

for all $i, j \in \{1, \ldots, r\}$, where the last equality follows from the Lipschitz-continuity of $\mathbf{M}$, which is a consequence of this property fulfilled by $\mathbf{Q}(\omega)$ and $\sum_{\nu=0}^{\infty} \mathbf{C}_\nu(p) e^{-i\nu\omega}$. Due to the formula $vec(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A}) vec(\mathbf{B})$ for appropriate matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ [cf. Lütkepohl (2005), p.662], the first term of the last right-hand side of (6.10) becomes

$$(\mathbf{M}(\omega) \otimes \overline{\mathbf{M}(\omega)}) vec\left(\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi} K(u)\left(\mathbf{I}_{n,\epsilon^+}(\omega - ub) - \frac{1}{2\pi}\mathbf{I}_r\right) du\right)$$

and it remains to show asymptotic normality for the part in big outer parentheses above. Plugging-in the expression $\mathbf{I}_{n,\epsilon^+}(\omega) = \frac{1}{2\pi}\sum_{s=-n+1}^{n-1} \widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) e^{-is\omega}$, where

$$\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-s} \underline{\epsilon}_{t+s}^+ \underline{\epsilon}_t^{+T}, & s \geq 0 \\ \frac{1}{n}\sum_{t=1-s}^{n} \underline{\epsilon}_{t+s}^+ \underline{\epsilon}_t^{+T}, & s < 0 \end{cases}, \tag{6.11}$$

we get

$$\begin{aligned} &\sqrt{nb}\frac{1}{2\pi}\int_{-\pi}^{\pi} K(u)\left(\mathbf{I}_{n,\epsilon^+}(\omega - ub) - \frac{1}{2\pi}\mathbf{I}_r\right) du \\ = \;& \sqrt{nb}\int_{-\pi}^{\pi} K(u)\frac{1}{2\pi}\sum_{s=1}^{n-1}\left(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) e^{-is(\omega-ub)} + \widehat{\mathbf{\Gamma}}_{\epsilon^+}(-s) e^{is(\omega-ub)}\right) du \\ &+\sqrt{nb}\frac{1}{2\pi}(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(0) - \mathbf{I}_r)\int_{-\pi}^{\pi} K(u) du, \end{aligned}$$

where the second term is $O_{P^*}(\sqrt{b}) = o_{P^*}(1)$. Using the fourier transform k of K and its symmetry, the first term can be written as

$$\frac{1}{\sqrt{2\pi}}\sqrt{nb}\sum_{s=1}^{n-1} k(sb)\left(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) e^{-is\omega} + \widehat{\mathbf{\Gamma}}_{\epsilon^+}(-s) e^{is\omega}\right).$$

Ignoring the factor $\frac{1}{\sqrt{2\pi}}$, we can split this expression to obtain

$$\sqrt{nb}\sum_{s=1}^{c_n} k(sb)\left(\widetilde{\mathbf{\Gamma}}_{\epsilon^+}(s) e^{-is\omega} + \widetilde{\mathbf{\Gamma}}_{\epsilon^+}(-s) e^{is\omega}\right) \tag{6.12}$$

$$+ \quad \sqrt{nb}\sum_{s=1}^{c_n} k(sb)\left(\left(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) - \widetilde{\mathbf{\Gamma}}_{\epsilon^+}(s)\right) e^{-is\omega} + \left(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(-s) - \widetilde{\mathbf{\Gamma}}_{\epsilon^+}(-s)\right) e^{is\omega}\right) \tag{6.13}$$

$$+ \quad \sqrt{nb}\sum_{s=c_n+1}^{n-1} k(sb)\left(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) e^{-is\omega} + \widehat{\mathbf{\Gamma}}_{\epsilon^+}(-s) e^{is\omega}\right), \tag{6.14}$$

where $(c_n : n \in \mathbb{N}) \subset \mathbb{N}$ satisfies $c_n = o(n)$ as well as $c_n \to \infty$ as $n \to \infty$ and the summation in $\widetilde{\mathbf{\Gamma}}_{\epsilon^+}(s)$ is from 1 to n compared to the definition of $\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s)$ in (6.11). Next we show that (6.13) and (6.14) vanish asymptotically. We prove this only for the parts with positive lags s. For the (h, j)-th component of (6.13) we get

$$\begin{aligned} E^+[|\sqrt{nb}\sum_{s=1}^{c_n} k(sb)\left(\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s) - \widetilde{\mathbf{\Gamma}}_{\epsilon^+}(s)\right)_{h,j} e^{-is\omega}|^2] \;&=\; nbE^+[|\sum_{s=1}^{c_n} k(sb)\frac{1}{n}\sum_{t=n-s+1}^{n} \epsilon_{t+s,h}^+ \epsilon_{t,j}^+ e^{-is\omega}|^2] \\ &\leq\; nb\sum_{s=1}^{c_n} k^2(sb)\frac{s}{n^2}. \end{aligned}$$

The last term is bounded by $\frac{c_n b}{n}\sum_{s=1}^{c_n} k^2(sb)$, which, defining $m_n = [\frac{1}{b}]$, is asymptotically equivalent to

$$\frac{c_n}{n}\frac{1}{m_n}\sum_{s=1}^{c_n} k^2(\frac{s}{m_n}) \;\cong\; \frac{c_n}{n}\int_0^{c_n} k^2(x)dx \to 0,$$

where $\int k(u)^2 du < \infty$ and $c_n = o(n)$ are used. Similarly, for the (h,j)-th component of (6.14), we get

$$\begin{aligned} E^+[|\sqrt{nb}\sum_{s=c_n+1}^{n-1} k(sb)\widehat{\mathbf{\Gamma}}_{\epsilon^+}(s)_{h,j}e^{-is\omega}|^2] &= nbE^+[|\sum_{s=c_n+1}^{n-1} k(sb)\frac{1}{n}\sum_{t=1}^{n-s}\underline{\epsilon}^+_{t+s,h}\underline{\epsilon}^+_{t,j}e^{-is\omega}|^2] \\ &\leq b\sum_{s=c_n+1}^{n-1} k^2(sb) \end{aligned}$$

and the last sum is asymptotically equivalent to

$$\frac{1}{m_n}\sum_{s=c_n+1}^{n-1} k^2(\frac{s}{m_n}) \;\cong\; \int_{c_n}^{\infty} k^2(x)dx \to 0.$$

Using expression (6.12), now, we define the quantity $\mathbf{W}^+_{t,n}$ by

$$\begin{aligned} \sqrt{nb}\sum_{s=1}^{c_n} k(sb)\left(\widetilde{\Gamma}_{\epsilon^+}(s)e^{-is\omega} + \widetilde{\Gamma}_{\epsilon^+}(-s)e^{is\omega}\right) &= \sum_{t=1}^{n}\sqrt{\frac{b}{n}}\sum_{s=1}^{c_n} k(sb)\left(\underline{\epsilon}^+_{t+s}\underline{\epsilon}^{+T}_t e^{-is\omega} + \underline{\epsilon}^+_{t-s}\underline{\epsilon}^{+T}_t e^{is\omega}\right) \\ &=: \sum_{t=1}^{n}\mathbf{W}^+_{t,n} \end{aligned}$$

and, by the Cramer-Wold device, finally, it remains to show asymptotic (complex) normality of $\sum_{t=1}^{n}\underline{c}^T vec\left(\mathbf{W}^+_{t,n}\right)$ for all $\underline{c} \in \mathbb{C}^{r^2}$, which per definition of the complex normal distribution is equivalent to asymptotic (real) normality of

$$\sum_{t=1}^{n}\underline{c}^T vec\left([Re(\mathbf{W}^+_{t,n})|Im(\mathbf{W}^+_{t,n})]\right) = \sum_{t=1}^{n}\underline{c}^{(1)T} vec\left(Re(\mathbf{W}^+_{t,n})\right) + \underline{c}^{(2)T} vec\left(Im(\mathbf{W}^+_{t,n})\right)$$

for all $\underline{c} = (\underline{c}^{(1)}, \underline{c}^{(2)})^T \in \mathbb{R}^{2r^2}$, where $Re(x)$ and $Im(x)$ denote the real and the imaginary part of a complex quantity x. These one-dimensional quantities can be treated standardly with Theorem 4 in Rosenblatt (1985, p.63) as done in Kreiß and Paparoditis (2003) for the univariate case to obtain asymptotic normality using the AAPB, which completes this proof. □

Lemma 6.4 (Bias term).
Assume (A1), (A2), (A3), (K1), (K3) and (B3). If $nb^5 \to C^2$ *with a constant* $C \geq 0$*, we get*

$$E^+[\widehat{\mathbf{f}}^*(\omega)] - \widetilde{\mathbf{f}}(\omega) \to \frac{C}{4\pi}\mathbf{f}''(\omega)\int K(u)u^2du$$

in probability, where $\mathbf{f}''(\omega)$ *is the (entrywise) second derivative in* ω *of the spectral density matrix* $\mathbf{f}$.

Proof.
Thanks to $|\frac{1}{n}\sum_{j=-N}^{N} K_b(\omega-\omega_j) - 1| = O(\frac{1}{nb})$ uniformly in ω and $E^+[\mathbf{I}^+_n(\omega_j)] = \widehat{\mathbf{f}}_{AR}(\omega_j)$, at

first, we get

$$\begin{aligned}&\sqrt{nb}(E^+[\widehat{\mathbf{f}}^*(\omega)]-\widetilde{\mathbf{f}}(\omega))\\=\;&\sqrt{nb}\left(\frac{1}{n}\sum_{j=-N}^{N}K_b(\omega-\omega_j)\left(\widetilde{\mathbf{Q}}(\omega_j)\widehat{\mathbf{f}}_{AR}(\omega_j)\overline{\widetilde{\mathbf{Q}}(\omega_j)}^T-\widetilde{\mathbf{Q}}(\omega)\widehat{\mathbf{f}}_{AR}(\omega)\overline{\widetilde{\mathbf{Q}}(\omega)}^T\right)\right)+O_P(\frac{1}{\sqrt{nb}}).\end{aligned}$$

Now, the expression in inner round parentheses can be displayed in the following way:

$$\begin{aligned}&\widetilde{\mathbf{Q}}(\omega_j)\widehat{\mathbf{f}}_{AR}(\omega_j)\overline{\widetilde{\mathbf{Q}}(\omega_j)}^T-\widetilde{\mathbf{Q}}(\omega)\widehat{\mathbf{f}}_{AR}(\omega)\overline{\widetilde{\mathbf{Q}}(\omega)}^T\\=\;&(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))\widehat{\mathbf{f}}_{AR}(\omega)\overline{\widetilde{\mathbf{Q}}(\omega)}^T+\widetilde{\mathbf{Q}}(\omega)(\widehat{\mathbf{f}}_{AR}(\omega_j)-\widehat{\mathbf{f}}_{AR}(\omega))\overline{\widetilde{\mathbf{Q}}(\omega)}^T\\&+\widetilde{\mathbf{Q}}(\omega)\widehat{\mathbf{f}}_{AR}(\omega)\overline{(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))}^T+(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))(\widehat{\mathbf{f}}_{AR}(\omega_j)-\widehat{\mathbf{f}}_{AR}(\omega))\overline{\widetilde{\mathbf{Q}}(\omega)}^T\\&+(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))\widehat{\mathbf{f}}_{AR}(\omega)\overline{(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))}^T+\widetilde{\mathbf{Q}}(\omega)(\widehat{\mathbf{f}}_{AR}(\omega_j)-\widehat{\mathbf{f}}_{AR}(\omega))\overline{(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))}^T\\&+(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))(\widehat{\mathbf{f}}_{AR}(\omega_j)-\widehat{\mathbf{f}}_{AR}(\omega))\overline{(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))}^T\\=\;&\widehat{\mathbf{D}}_{1,j}+\widehat{\mathbf{D}}_{2,j}+\widehat{\mathbf{D}}_{3,j}+\widehat{\mathbf{D}}_{4,j}+\widehat{\mathbf{D}}_{5,j}+\widehat{\mathbf{D}}_{6,j}+\widehat{\mathbf{D}}_{7,j},\end{aligned}$$

with an obvious notation for $\widehat{\mathbf{D}}_{k,j}$, $k=1,\ldots,7$. Note, because of the chain rule, for the second (componentwise) derivative of $\mathbf{f}(\omega)$, it holds

$$\begin{aligned}\mathbf{f}''(\omega)\;&=\;(\mathbf{Q}(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T)''\\&=\;\mathbf{Q}''(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T+\mathbf{Q}(\omega)\mathbf{f}''_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T+\mathbf{Q}(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}''(\omega)}^T\\&\quad+2\mathbf{Q}'(\omega)\mathbf{f}'_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T+2\mathbf{Q}'(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}'(\omega)}^T+2\mathbf{Q}(\omega)\mathbf{f}'_{AR}(\omega)\overline{\mathbf{Q}'(\omega)}^T\\&=\;\mathbf{D}_1+\mathbf{D}_2+\mathbf{D}_3+\mathbf{D}_4+\mathbf{D}_5+\mathbf{D}_6\end{aligned}$$

and the claimed convergence of $E^+[\widehat{\mathbf{f}}^*(\omega)]-\widetilde{\mathbf{f}}(\omega)$ follows from

$$\sqrt{\frac{b}{n}}\sum_{j=-N}^{N}K_b(\omega-\omega_j)\widehat{\mathbf{D}}_{k,j}\to\frac{C}{4\pi}\mathbf{D}_k\int K(u)u^2du,\quad k=1,\ldots,7 \tag{6.15}$$

in probability, where D_7 is set equal to zero. Consider first $\widehat{\mathbf{D}}_{1,j}$. A Taylor expansion of $\widetilde{Q}(\omega)$ delivers

$$\begin{aligned}&\sqrt{\frac{b}{n}}\sum_{j=-N}^{N}K_b(\omega-\omega_j)(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))\\=\;&\left(\sqrt{\frac{b}{n}}\sum_{j=-N}^{N}K_b(\omega-\omega_j)(\omega_j-\omega)\right)\widetilde{\mathbf{Q}}'(\omega)+\left(\frac{1}{2}\sqrt{\frac{b}{n}}\sum_{j=-N}^{N}K_b(\omega-\omega_j)(\omega_j-\omega)^2\right)\widetilde{\mathbf{Q}}''(\omega)\\&+\left(\frac{1}{6}\sqrt{\frac{b}{n}}\sum_{j=-N}^{N}K_b(\omega-\omega_j)(\omega_j-\omega)^3\widetilde{\mathbf{Q}}'''(\widetilde{\omega}_j)\right),\end{aligned}$$

with $\widetilde{\omega}_j$ between ω and ω_j. Due to $\int K(u)udu=0$ we get $\frac{1}{nb}\sum_{j=-N}^{N}K(\frac{\omega-\omega_j}{b})(\frac{\omega_j-\omega}{b})=O(\frac{1}{nb})$ and together with $nb^5=O(1)$ the first summand vanishes. The third is $O_P(b)$ because of $\widetilde{\mathbf{Q}}'''(\omega)=O_P(1)$ uniformly in ω and disappears also. From $nb^5\to C^2$ and Lemma 6.1, for the second term, we get

$$\left(\frac{\sqrt{nb^5}}{4\pi}\frac{2\pi}{nb}\sum_{j=-N}^{N}K(\frac{\omega-\omega_j}{b})(\frac{\omega_j-\omega}{b})^2\right)\widetilde{\mathbf{Q}}''(\omega)\to\frac{C}{4\pi}\mathbf{Q}''(\omega)\int K(u)u^2du,$$

which yields (6.15) for $k = 1$. The cases $k = 2$ and $k = 3$ can be treated analogously, where a Taylor expansion of $\widehat{\mathbf{f}}_{AR}(\omega)$ has to be used for $k = 2$. Now, consider $k \in \{4, 5, 6\}$. We prove only the case $k = 4$. Similar to calculations above, Taylor expansions of $\widetilde{\mathbf{Q}}(\omega)$ and $\widehat{\mathbf{f}}_{AR}(\omega)$, respectively, provide

$$\begin{aligned}
&\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))(\widehat{\mathbf{f}}_{AR}(\omega_j)-\widehat{\mathbf{f}}_{AR}(\omega)) \\
= &\left(\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\omega_j-\omega)^2\right)\widetilde{\mathbf{Q}}'(\omega)\widehat{\mathbf{f}}'_{AR}(\omega) \\
&\left(\frac{1}{2}\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\omega_j-\omega)^3\widetilde{\mathbf{Q}}''(\widetilde{\omega})\right)\widehat{\mathbf{f}}'_{AR}(\omega) \\
&+\widetilde{\mathbf{Q}}'(\omega)\left(\frac{1}{2}\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\omega_j-\omega)^3\widehat{\mathbf{f}}''_{AR}(\widetilde{\omega})\right) \\
&+\frac{1}{4}\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\omega_j-\omega)^4\widetilde{\mathbf{Q}}''(\widetilde{\omega})\widehat{\mathbf{f}}''_{AR}(\widetilde{\omega}) \\
\to\ &\frac{C}{4\pi}\mathbf{Q}'(\omega)\mathbf{f}'_{AR}(\omega)\int K(u)u^2du.
\end{aligned}$$

Finally, three times Taylor again yields

$$\begin{aligned}
&\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))(\widehat{\mathbf{f}}_{AR}(\omega_j)-\widehat{\mathbf{f}}_{AR}(\omega))\overline{(\widetilde{\mathbf{Q}}(\omega_j)-\widetilde{\mathbf{Q}}(\omega))}^T \\
= &\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\omega-\omega_j)(\omega_j-\omega)^3\widetilde{\mathbf{Q}}'(\widetilde{\omega})\widehat{\mathbf{f}}'_{AR}(\widetilde{\omega})\overline{\widetilde{\mathbf{Q}}'(\widetilde{\omega})}^T
\end{aligned}$$

and the last sum vanishes asymptotically, because of $\int K(u)u^3du = 0$. □

This concludes the proof of Theorem 5.2. □

6.4. **Autocovariances.**

Proof of Theorem 5.3.
Extending $\underline{X}_1^*, \ldots, \underline{X}_n^*$ cyclically to obtain $(\underline{X}_t^* : t \in \mathbb{Z})$, we can define

$$\widetilde{\mathbf{\Gamma}}^*(h) = \frac{1}{n}\sum_{t=1}^{n}(\underline{X}_{t+h} - \overline{\underline{X}})(\underline{X}_t - \overline{\underline{X}})^T, h \in \mathbb{Z}$$

and because of $E[\|\widehat{\mathbf{\Gamma}}^*(h) - \widetilde{\mathbf{\Gamma}}^*(h)\|] = O_P(\frac{1}{n})$ it suffices to show the assertion for the components of $\widetilde{\Gamma}^*(h)$. Let $h_1, h_2, k_1, k_2 \in \{1, \ldots, r\}$ as well as $h, k \in \mathbb{Z}$, then insertion and straightforward calculation yields

$$
\begin{aligned}
& nCov^+(\widetilde{\gamma}^*_{h_1h_2}(h) - E^+[\widetilde{\gamma}^*_{h_1h_2}(h)], \widetilde{\gamma}^*_{k_1k_2}(k) - E^+[\widetilde{\gamma}^*_{k_1k_2}(k)]) \\
= \; & \frac{4\pi^2}{n^3} \sum_{j_1,j_2,j_3,j_4=-N}^{N} \sum_{m_1,m_2,m_3,m_4=1}^{r} \widetilde{q}_{h_1m_1}(\omega_{j_1})\widetilde{q}_{h_2m_2}(\omega_{j_2})\widetilde{q}_{k_1m_3}(\omega_{j_3})\widetilde{q}_{k_2m_4}(\omega_{j_4}) \\
& \big(E^+[\underline{J}^+_{n,m_1}(\omega_{j_1})\underline{J}^+_{n,m_2}(\omega_{j_2})\underline{J}^+_{n,m_3}(\omega_{j_3})\underline{J}^+_{n,m_4}(\omega_{j_4})] \\
& -E^+[\underline{J}^+_{n,m_1}(\omega_{j_1})\underline{J}^+_{n,m_2}(\omega_{j_2})]E^+[\underline{J}^+_{n,m_3}(\omega_{j_3})\underline{J}^+_{n,m_4}(\omega_{j_4})]\big) \\
& e^{ih\omega_{j_1}} e^{ik\omega_{j_3}} \sum_{s=1}^{n} e^{is(\omega_{j_1}+\omega_{j_2})} \sum_{t=1}^{n} e^{it(\omega_{j_3}+\omega_{j_4})} \\
= \; & \frac{4\pi^2}{n} \sum_{j_1,j_2=-N}^{N} \sum_{m_1,m_2,m_3,m_4=1}^{r} \widetilde{q}_{h_1m_1}(\omega_{j_1})\overline{\widetilde{q}_{h_2m_2}(\omega_{j_1})}\widetilde{q}_{k_1m_3}(\omega_{j_2})\overline{\widetilde{q}_{k_2m_4}(\omega_{j_2})} \\
& \big(E^+[I^+_{n,m_1m_2}(\omega_{j_1})I^+_{n,m_3m_4}(\omega_{j_2})] - E^+[I^+_{n,m_1m_2}(\omega_{j_1})]E^+[I^+_{n,m_3m_4}(\omega_{j_2})]\big) \\
& e^{ih\omega_{j_1}} e^{ik\omega_{j_2}} \\
= \; & \frac{4\pi^2}{n} \sum_{j_1,j_2=-N}^{N} \sum_{m_1,m_2,m_3,m_4=1}^{r} \widetilde{q}_{h_1m_1}(\omega_{j_1})\overline{\widetilde{q}_{h_2m_2}(\omega_{j_1})}\widetilde{q}_{k_1m_3}(\omega_{j_2})\overline{\widetilde{q}_{k_2m_4}(\omega_{j_2})} \qquad (6.16) \\
& \frac{1}{4\pi^2n^2} \sum_{s,t,u,v=1}^{n} \sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty} \sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{r} \widehat{C}_{\nu_1,m_1\mu_1}(p)\widehat{C}_{\nu_2,m_2\mu_2}(p)\widehat{C}_{\nu_3,m_3\mu_3}(p)\widehat{C}_{\nu_4,m_4\mu_4}(p) \\
& \big(E^+[\epsilon^+_{s-\nu_1,\mu_1}\epsilon^+_{t-\nu_2,\mu_2}\epsilon^+_{u-\nu_3,\mu_3}\epsilon^+_{v-\nu_4,\mu_4}] - E^+[\epsilon^+_{s-\nu_1,\mu_1}\epsilon^+_{t-\nu_2,\mu_2}]E^+[\epsilon^+_{u-\nu_3,\mu_3}\epsilon^+_{v-\nu_4,\mu_4}]\big) \\
& e^{-i(s-t)\omega_{j_1}} e^{-i(u-v)\omega_{j_2}} e^{ih\omega_{j_1}} e^{ik\omega_{j_2}}.
\end{aligned}
$$

For the second equality from above we used $\sum_{t=1}^{n} e^{it\omega} = 0$ if $\omega \neq 0$ and n otherwise as soon as the hermitian symmetry of $\underline{J}_n(\omega)$ and $\widetilde{\mathbf{Q}}(\omega)$. Inserting for the periodogram provides the third equation. Thanks to

$$
\begin{aligned}
& \sum_{s,t,u,v=1}^{n} \big(E[\epsilon^+_{s-\nu_1,\mu_1}\epsilon^+_{t-\nu_2,\mu_2}\epsilon^+_{u-\nu_3,\mu_3}\epsilon^+_{v-\nu_4,\mu_4}] - E[\epsilon^+_{s-\nu_1,\mu_1}\epsilon^+_{t-\nu_2,\mu_2}]E[\epsilon^+_{u-\nu_3,\mu_3}\epsilon^+_{v-\nu_4,\mu_4}]\big) \\
& e^{-i(s-t)\omega_{j_1}} e^{-i(u-v)\omega_{j_2}} \\
= \; & n\widehat{\kappa}_4(p;\mu_1,\mu_2,\mu_3,\mu_4)e^{-i\nu_1\omega_{j_1}} e^{i\nu_2\omega_{j_1}} e^{-i\nu_3\omega_{j_2}} e^{i\nu_4\omega_{j_2}} \qquad (6.17) \\
& +\widehat{\sigma}_{\mu_1\mu_3}(p)\widehat{\sigma}_{\mu_2\mu_4}(p) \left|\sum_{s=1}^{n} e^{-is(\omega_{j_1}+\omega_{j_2})}\right|^2 e^{i\nu_1\omega_{j_2}} e^{-i\nu_2\omega_{j_2}} e^{-i\nu_3\omega_{j_2}} e^{i\nu_4\omega_{j_2}} \qquad (6.18) \\
& +\widehat{\sigma}_{\mu_1\mu_4}(p)\widehat{\sigma}_{\mu_2\mu_3}(p) \left|\sum_{s=1}^{n} e^{-is(\omega_{j_1}-\omega_{j_2})}\right|^2 e^{-i\nu_1\omega_{j_2}} e^{i\nu_2\omega_{j_2}} e^{-i\nu_3\omega_{j_2}} e^{i\nu_4\omega_{j_2}}, \qquad (6.19)
\end{aligned}
$$

where $\widehat{\kappa}_4(p;\mu_1,\mu_2,\mu_3,\mu_4)$ denotes the fourth-order cumulant between the residuals $\epsilon^+_{t,\mu_1}, \epsilon^+_{t,\mu_2}, \epsilon^+_{t,\mu_3}$ and ϵ^+_{t,μ_4} obtained by fitting an AR-model of order p, insertion in (6.16) simplifies matters and we have to deal with the three summands in (6.17)-(6.19) separately. Consider first (6.18). Here, the sum over j_2 in (6.16) collapses and a rearrangement results in

$$\begin{aligned}
&\frac{4\pi^2}{n}\sum_{j_1=-N}^{N}\sum_{m_1,m_3=1}^{r}\widetilde{q}_{h_1m_1}(\omega_{j_1})\\
&\left(\frac{1}{2\pi}\sum_{\mu_1,\mu_3=1}^{r}\left(\sum_{\nu_1=0}^{\infty}\widehat{C}_{\nu_1,m_1\mu_1}(p)e^{-i\nu_1\omega_{j_1}}\right)\widehat{\sigma}_{\mu_1\mu_3}(p)\overline{\left(\sum_{\nu_3=0}^{\infty}\widehat{C}_{\nu_3,\mu_3m_3}(p)e^{-i\nu_3\omega_{j_1}}\right)}^T\right)\\
&\overline{\widetilde{q}_{m_3k_1}(\omega_{j_1})}^T\sum_{m_2,m_4=1}^{r}\widetilde{q}_{k_2m_4}(\omega_{j_1})\\
&\left(\frac{1}{2\pi}\sum_{\mu_4,\mu_2=1}^{r}\left(\sum_{\nu_4=0}^{\infty}\widehat{C}_{\nu_4,m_4\mu_4}(p)e^{-i\nu_4\omega_{j_1}}\right)\widehat{\sigma}_{\mu_4\mu_2}(p)\overline{\left(\sum_{\nu_2=0}^{\infty}\widehat{C}_{\nu_2,\mu_2m_2}(p)e^{-i\nu_2\omega_{j_1}}\right)}^T\right)\\
&\overline{\widetilde{q}_{m_2h_2}(\omega_{j_1})}^T e^{-i(k-h)\omega_{j_1}}\\
=\ &\frac{4\pi^2}{n}\sum_{j_1=-N}^{N}\left(\widetilde{\mathbf{Q}}(\omega_{j_1})\widehat{\mathbf{f}}_{AR}(\omega_{j_1})\overline{\widetilde{\mathbf{Q}}(\omega_{j_1})}^T\right)_{h_1k_1}\left(\widetilde{\mathbf{Q}}(\omega_{j_1})\widehat{\mathbf{f}}_{AR}(\omega_{j_1})\overline{\widetilde{\mathbf{Q}}(\omega_{j_1})}^T\right)_{k_2h_2}e^{-i(k-h)\omega_{j_1}}.
\end{aligned}$$

Because of the uniform convergence in ω of the quantities $\widetilde{\mathbf{Q}}(\omega)$ and $\widehat{\mathbf{f}}_{AR}(\omega)$, the Riemann sum above converges to

$$\begin{aligned}
&2\pi\int_{-\pi}^{\pi}\left(\mathbf{Q}(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T\right)_{h_1k_1}\left(\mathbf{Q}(\omega)\mathbf{f}_{AR}(\omega)\overline{\mathbf{Q}(\omega)}^T\right)_{k_2h_2}e^{-i(k-h)\omega}d\omega\\
=\ &2\pi\int_{-\pi}^{\pi}f_{h_1k_1}(\omega)f_{k_2h_2}(\omega)e^{-i(k-h)\omega}d\omega
\end{aligned}$$

in probability. Finally, the multivariate inversion formula yields

$$\begin{aligned}
2\pi\int_{-\pi}^{\pi}f_{h_1k_1}(\omega)f_{k_2h_2}(\omega)e^{-i(k-h)\omega}d\omega &= 2\pi\int_{-\pi}^{\pi}f_{h_1k_1}(\omega)\frac{1}{2\pi}\sum_{t=-\infty}^{\infty}e^{-it\omega}\gamma_{k_2h_2}(t)e^{-i(k-h)\omega}d\omega\\
&= \sum_{t=-\infty}^{\infty}\gamma_{k_2h_2}(t)\int_{-\pi}^{\pi}f_{h_1k_1}(\omega)e^{i(-t-k+h)\omega}d\omega\\
&= \sum_{t=-\infty}^{\infty}\gamma_{h_2k_2}(-t)\gamma_{h_1k_1}(-t-k+h).
\end{aligned}$$

Arguments are analoguous for (6.19) and its limit in probability is

$$\sum_{t=-\infty}^{\infty}\gamma_{h_2k_2}(-t)\gamma_{h_1k_1}(-t-k+h).$$

It remains to check (6.17). Inserting in (6.16) and rearranging gives the expression

$$\sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{r} \left(\frac{2\pi}{n} \sum_{j_1=-N}^{N} \left(\widetilde{\mathbf{Q}}(\omega_{j_1}) \left(\frac{1}{\sqrt{2\pi}} \sum_{\nu_1=0}^{\infty} \widehat{\mathbf{C}}_{\nu_1}(p) e^{-i\nu_1\omega_{j_1}} \right) \right)_{h_1\mu_1} \right.$$
$$\left(\left(\overline{\left(\frac{1}{\sqrt{2\pi}} \sum_{\nu_2=0}^{\infty} \widehat{\mathbf{C}}_{\nu_2}(p) e^{-i\nu_2\omega_{j_1}} \right)}^T \overline{\widetilde{\mathbf{Q}}(\omega_{j_1})}^T \right)_{\mu_2 h_2} e^{ih\omega_{j_1}} \right)$$
$$\widehat{\kappa}_4(p;\mu_1,\mu_2,\mu_3,\mu_4)$$
$$\left(\frac{2\pi}{n} \sum_{j_2=-N}^{N} \left(\widetilde{\mathbf{Q}}(\omega_{j_2}) \left(\frac{1}{\sqrt{2\pi}} \sum_{\nu_3=0}^{\infty} \widehat{\mathbf{C}}_{\nu_3}(p) e^{-i\nu_3\omega_{j_2}} \right) \right)_{k_1\mu_3} \right.$$
$$\left. \left(\left(\overline{\left(\frac{1}{\sqrt{2\pi}} \sum_{\nu_4=0}^{\infty} \widehat{\mathbf{C}}_{\nu_4}(p) e^{-i\nu_4\omega_{j_2}} \right)}^T \overline{\widetilde{\mathbf{Q}}(\omega_{j_2})}^T \right)_{\mu_4 k_2} e^{ik\omega_{j_2}} \right) \right),$$

which converges to the corresponding part as stated in the theorem. □

Proof of Theorem 5.4.
As in the proof of Theorem 5.2 it is more convenient to show asymptotic normality for

$$\mathcal{L}\left(\sqrt{nb}vec\left(\left[\widehat{\mathbf{\Gamma}}^*(0) - E^+[\widehat{\mathbf{\Gamma}}^*(0)]|\cdots|\widehat{\mathbf{\Gamma}}^*(s) - E^+[\widehat{\mathbf{\Gamma}}^*(s)]\right]\right)|\underline{X}_1,\ldots,\underline{X}_n\right)$$

and analogue to the proof of Lemma 6.3 it suffices here to consider the case $s = 1$ with some lag h. Hence, we can focus on

$$\sqrt{nb}vec\left(\widehat{\mathbf{\Gamma}}^*(h) - E^+[\widehat{\mathbf{\Gamma}}^*(h)]\right).$$

Recall that $\widehat{\mathbf{\Gamma}}^*(h)$ can be displayed as a so-called spectral mean [cf. Dahlhaus (1985) for the univariate case], that is

$$\widehat{\mathbf{\Gamma}}^*(h) = \int_{\pi}^{\pi} \mathbf{I}_n^*(\omega) e^{ih\omega} d\omega.$$

Using $\mathbf{I}_n^*(\omega) = \widetilde{\mathbf{Q}}(\omega)\mathbf{I}_n^+(\omega)\overline{\widetilde{\mathbf{Q}}(\omega)}^T$ and $\mathbf{I}_n^+(\omega) = \frac{1}{2\pi}\sum_{k=-(n-1)}^{n-1} \widehat{\mathbf{\Gamma}}^+(k)e^{-ik\omega}$, where $\widehat{\mathbf{\Gamma}}^+(h)$ is analogue to (5.1) based on $\underline{X}_1^+,\ldots,\underline{X}_n^+$, we get

$$\widehat{\mathbf{\Gamma}}^*(h) = \sum_{k=-(n-1)}^{n-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \widetilde{\mathbf{Q}}(\omega)\widehat{\mathbf{\Gamma}}^+(k)\overline{\widetilde{\mathbf{Q}}(\omega)}^T e^{-i(k-h)\omega} d\omega.$$

Further, due to the formula $vec(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})vec(\mathbf{B})$ for appropriate matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, application of the vec-operator yields

$$\begin{aligned} &\sqrt{n}vec(\widehat{\mathbf{\Gamma}}^*(h) - E^+[\widehat{\mathbf{\Gamma}}^*(h)]) \\ = &\sum_{k=-(n-1)}^{n-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\overline{\widetilde{\mathbf{Q}}(\omega)} \otimes \widetilde{\mathbf{Q}}(\omega)\right) e^{-i(k-h)\omega} d\omega \sqrt{n}\ vec\left(\widehat{\mathbf{\Gamma}}^+(k) - E^+[\widehat{\mathbf{\Gamma}}^+(k)]\right) \\ = &\underline{Z}_n^+. \end{aligned}$$

To make Proposition 6.3.9 in Brockwell and Davis (1991) applicable, let $M \in \mathbb{N}$ be fixed and split the last sum in two parts to obtain

$$\begin{aligned}
&\sum_{k=-M}^{M} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(k-h)\omega}d\omega\sqrt{n}\ vec\left(\widehat{\mathbf{\Gamma}}^{+}(k)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(k)]\right)\\
&+\sum_{\substack{k=-(n-1)\\|k|>M}}^{n-1} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(k-h)\omega}d\omega\sqrt{n}\ vec\left(\widehat{\mathbf{\Gamma}}^{+}(k)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(k)]\right),\\
=\ & \underline{Z}^{+}_{n,M}+(\underline{Z}^{+}_{n,M}-\underline{Z}^{+}_{n}),
\end{aligned}$$

with an obvious notation for $\underline{Z}^{+}_{n,M}$. Now, it suffices to have that for all $M \in \mathbb{N}$ the quantity $\underline{Z}^{+}_{n,M}$ converges weakly to a normal distribution in probability depending on M, which itself in turn converges for $M \to \infty$. Moreover, for all $\epsilon > 0$, the condition

$$\lim_{M\to\infty}\limsup_{n\to\infty} P^{+}(\|\underline{Z}_n-\underline{Z}_{n,M}\|>\epsilon)=0 \tag{6.20}$$

in probability has to be satisfied. At first, let M be fixed. Then $\underline{Z}^{+}_{n,M}$ can be displayed as a matrix-vector product and we get

$$\begin{aligned}
\underline{Z}^{+}_{n,M} &= \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(-M-h)\omega}d\omega\right|\cdots\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(M-h)\omega}d\omega\right]\\
&\quad\cdot\sqrt{n}vec\left(\left[\widehat{\mathbf{\Gamma}}^{+}(-M)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(-M)]\right|\cdots\left|\widehat{\mathbf{\Gamma}}^{+}(M)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(M)]\right]\right)\\
&= \mathbf{H}^{+}_{n,M}\cdot\underline{R}^{+}_{n,M},
\end{aligned}$$

where the $(r^2 \times (2n-1)r^2)$-matrix $\mathbf{H}^{+}_{n,M}$ is multiplied with the $(2n-1)r^2$-dimensional vector $\underline{R}^{+}_{n,M}$. Applying an adequate CLT (e.g. the CLT in Neumann and Paparoditis (2008)), we get asymptotic normality of $\underline{R}^{+}_{n,M}$, which contains nothing else but empirical autocovariances of the usual residual AR-bootstrap. Together with the convergence in probability of $\mathbf{H}^{+}_{n,M}$ and Slutsky we get the required weak convergence of $\underline{Z}^{+}_{n,M} = H^{+}_{n,M}\cdot\underline{R}^{+}_{n,M}$ and its asymptotic multivariate normal distribution depending on M converges itself to the correct covariance matrix as $M \to \infty$ by Theorem 5.3. It remains to show (6.20). By Markov inequality, it suffices to consider

$$\begin{aligned}
&E^{+}[\|\underline{Z}_n-\underline{Z}_{n,M}\|]\\
=\ &\|\sum_{\substack{k=-(n-1)\\|k|>M}}^{n-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(k-h)\omega}d\omega\sqrt{n}\ vec\left(\widehat{\mathbf{\Gamma}}^{+}(k)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(k)]\right)\|\\
\leq\ &\sum_{\substack{k=-(n-1)\\|k|>M}}^{n-1}\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(k-h)\omega}d\omega\|\cdot\|\sqrt{n}\ vec\left(\widehat{\mathbf{\Gamma}}^{+}(k)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(k)]\right)\|\\
=\ &O_P(1)\sum_{\substack{k=-(n-1)\\|k|>M}}^{n-1}\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\overline{\widetilde{\mathbf{Q}}(\omega)}\otimes\widetilde{\mathbf{Q}}(\omega)\right)e^{-i(k-h)\omega}d\omega\|,
\end{aligned}\tag{6.21}$$

where we have used $\sqrt{n}\ vec(\widehat{\mathbf{\Gamma}}^{+}(k)-E^{+}[\widehat{\mathbf{\Gamma}}^{+}(k)]) = O_P(1)$ uniformly in k. Now, let $\|\cdot\|$ be the 1-norm for matrices $\|\cdot\|_1$, defined as $\|\mathbf{A}\|_1 = \sum_{i,j}|a_{i,j}|$. The normed expression in (6.21) is a matrix, whose entries are usual fourier coefficients of the type

$$a_{k-h}(r,s,t,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\widetilde{q}_{rs}(\omega)}\widetilde{q}_{tu}(\omega)e^{-i(k-h)\omega}d\omega,$$

where $r,s,t,u \in \{1,\ldots,r\}$. Because of Lemma 6.1, the function $\overline{\widetilde{q}_{rs}(\cdot)}\widetilde{q}_{tu}(\cdot)$ is three times differentiable and therefore $|a_{k-h}|$ can be bounded by $\frac{T_n}{|k-h|^2}$, where

$$T_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\partial^2}{\partial\omega^2}\left(\overline{\widetilde{q}_{rs}(\omega)}\widetilde{q}_{tu}(\omega)\right)\right| d\omega = O_P(1)$$

uniformly in k. Finally, for sufficiently large M, we obtain

$$\lim_{M\to\infty}\limsup_{n\to\infty} E^+[\|\underline{Z}_n - \underline{Z}_{n,M}\|] \quad\leq\quad \lim_{M\to\infty}\limsup_{n\to\infty} O_P(1)\sum_{\substack{k=-(n-1)\\|k|>M}}^{n-1}\frac{1}{|k-h|^2} = 0$$

in probability, where we have used $\sum_{|k|>M}\frac{1}{|k-h|^2} < \infty$. This concludes the proof. □

Proof of Corollary 5.4.
All we have to show is that the asymptotic covariance derived in Theorem 5.3 agrees with (5.2). It suffices to consider the first part containing the fourth order cumulants, because the second parts are already equal. Note, that the sums over ν_1 and ν_2 in (5.2) are from 0 to ∞ due to causality. Under the assumption of an underlying $VAR(p_0)$-model, fitting a $VAR(p)$-model with $p \geq p_0$, we estimate the parameters $\mathbf{A}_1,\ldots,\mathbf{A}_{p_0}$ consistently with $\widehat{\mathbf{A}}_1(p),\ldots,\widehat{\mathbf{A}}_p(p)$, where $\widehat{\mathbf{A}}_k(p)$ converges to zero for $k > p_0$. Thus, we obtain $\mathbf{C}_\nu(p) = \mathbf{C}_\nu$ for all $\nu \in \mathbb{N}_0$ on the one hand and $\mathbf{f} = \mathbf{f}_{AR}$ on the other hand, which in turn yields $\widetilde{\mathbf{Q}}(\omega) \to \mathbf{Q}(\omega) = \mathbf{I}_r$ in probability uniformly in ω. Moreover, it holds $\kappa_4(p;\mu_1,\mu_2,\mu_3,\mu_4) = \kappa_4(\mu_1,\mu_2,\mu_3,\mu_4)$. Together the first part of expression (5.3) becomes

$$\sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{r}\left(\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2\pi}}\sum_{\nu_1=0}^{\infty}\mathbf{C}_{\nu_1}e^{-i\nu_1\omega_1}\right)_{h_1\mu_1}\left(\overline{\left(\frac{1}{\sqrt{2\pi}}\sum_{\nu_2=0}^{\infty}\mathbf{C}_{\nu_2}e^{-i\nu_2\omega_1}\right)}^T\right)_{\mu_2h_2}e^{ih\omega_1}d\omega_1\right)$$
$$\kappa_4(\mu_1,\mu_2,\mu_3,\mu_4)$$
$$\left(\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2\pi}}\sum_{\nu_3=0}^{\infty}\mathbf{C}_{\nu_3}e^{-i\nu_3\omega_2}\right)_{k_1\mu_3}\left(\overline{\left(\frac{1}{\sqrt{2\pi}}\sum_{\nu_4=0}^{\infty}\mathbf{C}_{\nu_4}e^{-i\nu_4\omega_2}\right)}^T\right)_{\mu_4k_2}e^{ik\omega_2}d\omega_2\right)$$

and the first (analogue for the second) integral above is equal to

$$\sum_{\nu_1,\nu_2=0}^{\infty}C_{\nu_1,h_1\mu_1}C_{\nu_2,h_2\mu_2}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(\nu_1-\nu_2-h)\omega_1}d\omega_1 = \sum_{\nu_1=0}^{\infty}C_{\nu_1,h_1\mu_1}C_{\nu_1-h,h_2\mu_2}.$$

An analogue calculation concludes the proof. □

7. A simulation study

In this section we compare the performance of the proposed multiple hybrid bootstrap to that of the usual autoregressive bootstrap and that of the moving block bootstrap by means of simulation. In order to make such a comparison, we have chosen statistics for which all methods lead to asymptotically correct approximations. In particular, we study and compare the performance of the aforementioned bootstrap methods in estimating a) the variance σ^2 of the first component and b) the covariance γ_{12} of both components of the sample mean $\overline{\underline{X}} = \frac{1}{n}\sum_{t=1}^{n}\underline{X}_t$ of a bivariate time series data set.

Realizations of length $n = 50$ and $n = 400$ from two models

$$\underline{X}_t = \mathbf{A}_1 \underline{\epsilon}_{t-1} + \underline{\epsilon}_t \qquad \text{and} \qquad \underline{X}_t = \mathbf{A}_1 \underline{X}_{t-1} + \underline{\epsilon}_t$$

with i.i.d. $\underline{\epsilon}_t \sim \mathcal{N}(\underline{0}, \mathbf{\Sigma})$ have been considered, where the first one is a vector moving average model of order one ($VMA(1)$-model) and the second is a vector autoregressive model of order one ($VAR(1)$-model). In both cases, we have used

$$\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.9 \\ 0.0 & 0.5 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{pmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \end{pmatrix}.$$

To estimate the exact variance σ^2 and covariance γ_{12}, $10,000$ Monte-Carlo replications have been used while the bootstrap approximations are based on $B = 300$ bootstrap replications and we have simulated $M = 200$ data sets, respectively. In all cases, the Bartlett-Priestley kernel K has been used and an autoregressive model of order $p = 1$ is fitted to the data.

In Figure 1-4, some boxplots of the distributions of the different bootstrap approximations for the cases $n = 50$ and $n = 400$ are presented. To check how sensitive the hybrid bootstrap reacts concerning the choice of the bandwidth h in Figure 5 and 6 boxplots with different bandwidths are shown.

All figures show reasonable results for the hybrid bootstrap in comparison to the other methods, but the effect of the nonparametric correction is clearly seen in Figure 2, where the bias of the pure autoregressive bootstrap is reduced significantly. Moreover, as expected, the hybrid bootstrap works well for autoregressive time series data as illustrated in Figure 3 and 4, where even some bias reduction can be seen in comparison to the autoregressive bootstrap. The Figures 5 and 6 demonstrate that the hybrid bootstrap seems not to be over sensitive concerning the choice of h. In particular, the right panel in Figure 5 shows the typical behaviour of decreasing fluctuation with increasing bandwidth.

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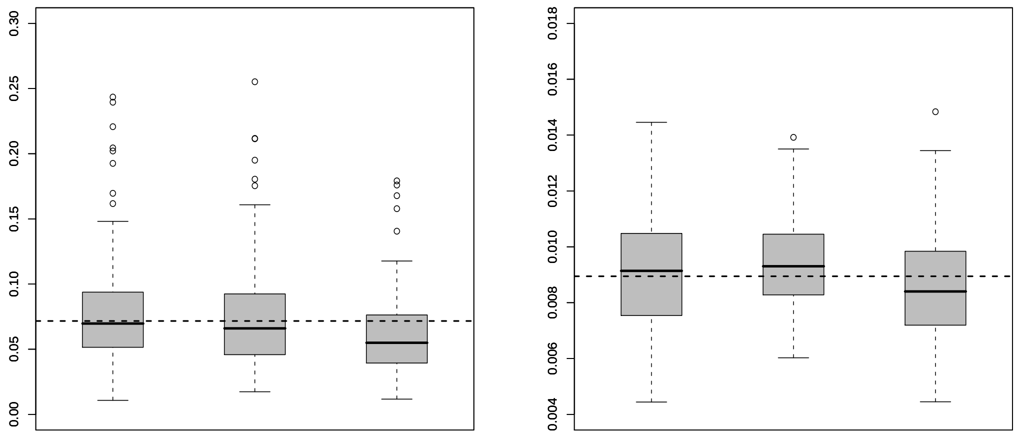

FIGURE 1. Boxplots of the bootstrap distributions for the variance of the first component of the sample mean in the $VMA(1)$-case with target indicated by the horizontal dashed line. In both panels from left to right: hybrid bootstrap (HB), AR-bootstrap (ARB) and moving block bootstrap (MBB). Left panel: $n = 50$, HB with $p = 1$ and $h = 0,3$; ARB with $p = 1$; MBB with $l = 5$. Right panel: $n = 400$, HB with $p = 1$ and $h = 0,15$; ARB with $p = 1$; MBB with $l = 10$.

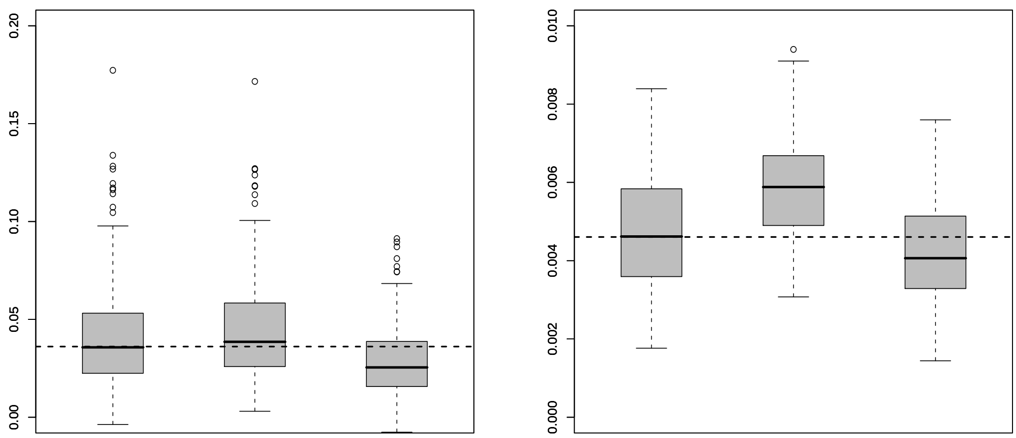

FIGURE 2. Boxplots of the bootstrap distributions for the covariance of both components of the sample mean in the $VMA(1)$-case with target indicated by the horizontal dashed line. In both panels from left to right: hybrid bootstrap (HB), AR-bootstrap (ARB) and moving block bootstrap (MBB). Left panel: $n = 50$, HB with $p = 1$ and $h = 0,3$; ARB with $p = 1$; MBB with $l = 5$. Right panel: $n = 400$, HB with $p = 1$ and $h = 0,15$; ARB with $p = 1$; MBB with $l = 10$.

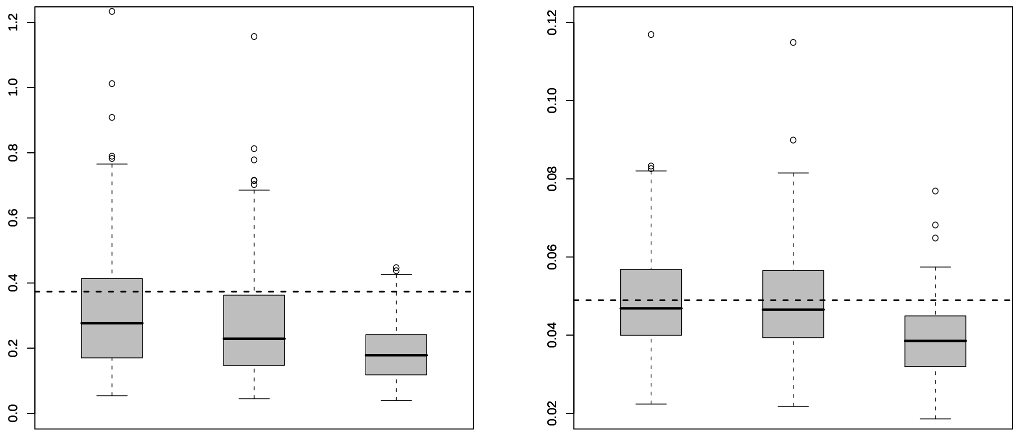

Figure 3. Boxplots of the bootstrap distributions for the variance of the first component of the sample mean in the $VAR(1)$-case with target indicated by the horizontal dashed line. In both panels from left to right: hybrid bootstrap (HB), AR-bootstrap (ARB) and moving block bootstrap (MBB). Left panel: $n = 50$, HB with $p = 1$ and $h = 0,3$; ARB with $p = 1$; MBB with $l = 5$. Right panel: $n = 400$, HB with $p = 1$ and $h = 0,15$; ARB with $p = 1$; MBB with $l = 10$.

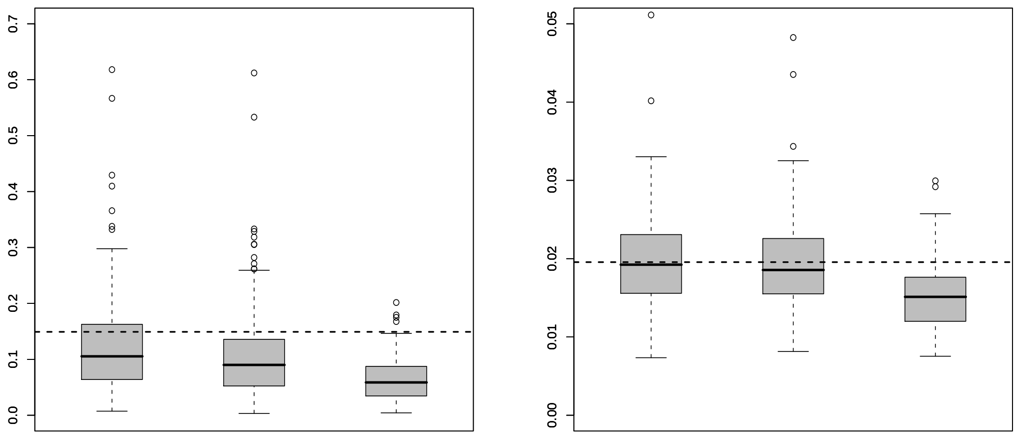

Figure 4. Boxplots of the bootstrap distributions for the covariance of both components of the sample mean in the $VAR(1)$-case with target indicated by the horizontal dashed line. In both panels from left to right: hybrid bootstrap (HB), AR-bootstrap (ARB) and moving block bootstrap (MBB). Left panel: $n = 50$, HB with $p = 1$ and $h = 0,3$; ARB with $p = 1$; MBB with $l = 5$. Right panel: $n = 400$, HB with $p = 1$ and $h = 0,15$; ARB with $p = 1$; MBB with $l = 10$.

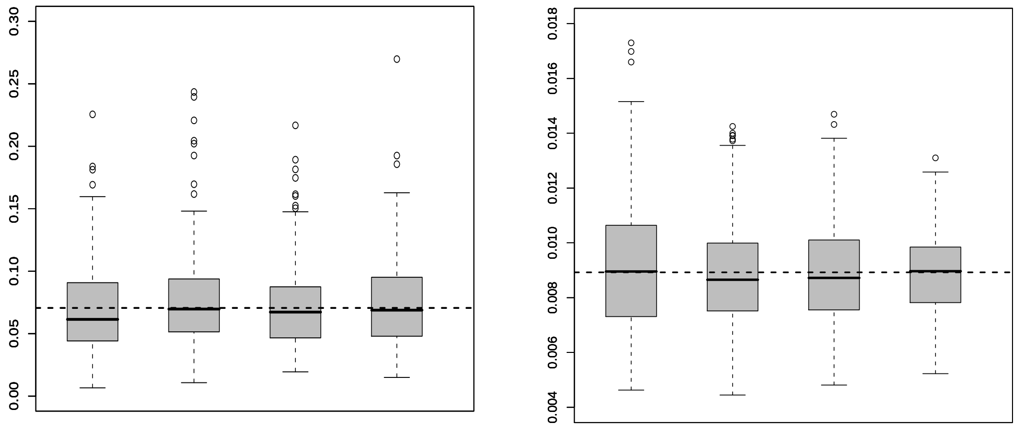

FIGURE 5. Boxplots of the bootstrap distributions for the variance of the first component of the sample mean using hybrid bootstrap (HB) in the $VMA(1)$-case with target indicated by the horizontal dashed line for different bandwidths h. Left panel: $n = 50$, from left to right: $h = 0.2$, $h = 0.3$, $h = 0.4$ and $h = 0.5$. Right panel: $n = 400$, from left to right: $h = 0.1$, $h = 0.15$, $h = 0.2$ and $h = 0.25$.

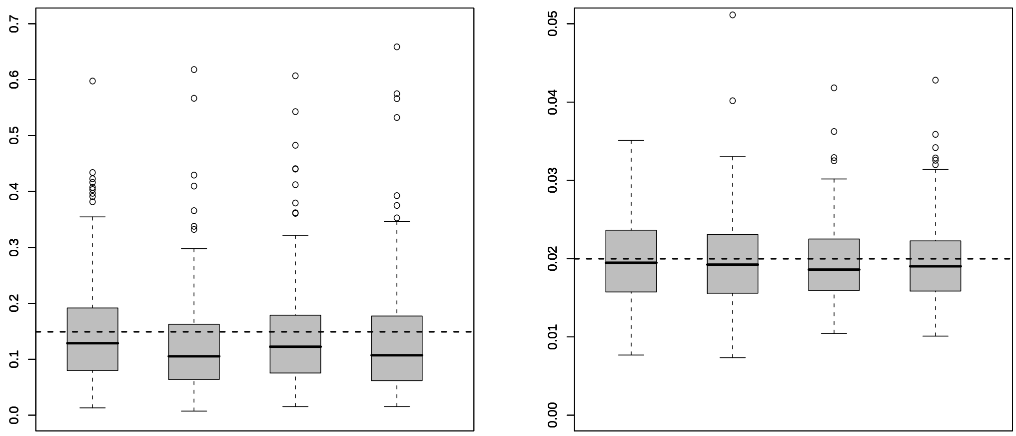

FIGURE 6. Boxplots of the bootstrap distributions for the covariance of both components of the sample mean using hybrid bootstrap (HB) in the $VAR(1)$-case with target indicated by the horizontal dashed line for different bandwidths h. Left panel: $n = 50$, from left to right: $h = 0.2$, $h = 0.3$, $h = 0.4$ and $h = 0.5$. Right panel: $n = 400$, from left to right: $h = 0.1$, $h = 0.15$, $h = 0.2$ and $h = 0.25$.